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Localized axial Green's function method for the convection–diffusion equations in arbitrary domains

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ABSTRACT

A localized axial Green's function method (LAGM) is proposed for the convection–diffusion equation. The axial Green's function method (AGM) enables us to calculate the numerical solution of a multi-dimensional problem using only one-dimensional Green's functions for the axially split differential operators. This AGM has been developed not only for the elliptic boundary value problems but also for the steady Stokes flows, however, this paper is concerned with the localization of the AGM. This localization of the method is needed for practical purpose when computing the axial Green's function, specifically for the convection–diffusion equation on a line segment that we call the local axial line. Although our focus is mainly on the convection-dominated cases in arbitrary domains, this method can solve other cases in a unified way. Numerical results show that, despite irregular types of discretization on an arbitrary domain, we can calculate the numerical solutions using the LAGM without loss of accuracy even in cases of large convection. In particular, it is also shown that randomly distributed axial lines are available in our LAGM and complicated domains are not a burden.

1. Introduction

In order to efficiently and accurately solve the convection–diffusion problems in arbitrary domains, we propose an innovative approach compared to other methods: the *localized* axial Green's function method (LAGM). This method stems from the axial Green's function method (AGM) [1,2] which we have developed for solving the elliptic boundary value problems [1] as well as the Stokes flows in complicated domains [2]. The AGM is first developed for solving a general class of elliptic problems in complex geometry only using one-dimensional Green's functions of analytic form on the straight lines along axes maximally belonging to the problem domain. The maximality of the straight lines means that both the starting and ending points of the line are located on the boundary of the problem domain. These lines are called the axial lines, for example, x-axial lines and y-axial lines in 2D.

In cases where the diffusion coefficient becomes small, the convection–diffusion equation tends toward the convection-dominated case. This makes the problem hard to solve numerically because of the interior and/or boundary layers. The convergence feature in many numerical methods to solve these kinds of problems usually depends on the ratio between the convection and diffusion terms, called the Péclet number. Additionally, the arbitrary domains often cause another difficulty when developing accurate numerical methods. The irregular type of grid or mesh for discretization in an arbitrary domain

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usually degrades the accuracy of numerical solutions. The purpose of this paper is mainly focused on developing a practical, available numerical method in arbitrary domains without loss of any accuracy of its numerical solution. The original AGM looks similar to the boundary element method, but in reality they are very different, while the LAGM has a close relationship to the finite difference method on the uniform grid.

There are many modern methods as investigated by Augustine et al. [3] where the authors have reported that the modern approaches do not seem to be more beneficial than the classical finite volume method (FVM) and the Streamlined-Upwind Petrov–Galerkin (SUPG) scheme. The SUPG is developed as a stabilization scheme in the finite element method in order to solve the convection-dominated problems. The convergence rate of the SUPG is of $O(h^{\frac{3}{2}})$ in L^2-sense theoretically, which is not optimal [4]. Nguyen et al. [4] infer that adaptive mesh algorithms enable us to improve this convergence rate. It is also pointed out by Augustin et al. [3] that an appropriate grid or mesh in FVM might be essential to avoid smearing in the steep layers. Other variants of SUPG have been proposed by Delsaute et al. [5] by selecting particular test functions different from the standard SUPG. However, the quality mesh or body-fitted grid generation is in fact burdensome in arbitrary domains. The extended finite element method (XFEM) can be a candidate for the high gradient solution in complicated domains but it has to employ a smoothing step function with some parameters properly determined [6].

To conduct higher-order accurate schemes beyond the lower-order FVM, Cockburn and Shu [7] have developed the Runge–Kutta discontinuous Galerkin method, which turns out to be a generalization of the standard FVM. Afif et al. [8] proved the convergence of an FVM equipped with the Godunov scheme for the convection term, and also showed the discrete maximum principle. It is also reported that, in the stabilized Galerkin scheme under the acute-angled triangulations, there is some relationship between the discrete maximum principle and the accuracy of the numerical solutions [9].

Based on the asymptotic analysis, Chapman et al. [10] and Romero [11] solve problems of a high Péclet number, respectively. From a theoretical point of view, Novikov et al. [12] investigated the behavior of boundary layer solutions in the water pipe problem at a high Péclet number. From a practical viewpoint, Meent et al. [13] have studied the biological problem of a high Péclet number, at which a rotational cytoplasmic streaming inside the plant Chara occurs. More comprehensive problems have been well addressed in the review [14] written by Shu, in which high order WENO schemes are used for convection-dominated cases.

When using the boundary element methods (BEM) [15–17] in convection–diffusion problems, the accuracy of numerical solutions strongly depends on which numerical integration schemes are adopted. One advantage of BEM is that no artificial up-winding is necessary. Among these methods, Qiu et al. [15] have pointed out that, as the Péclet number increases, the system matrix becomes sparser and more diagonal dominant. However, the fundamental solution or Green's function in high-dimensional space for the convection–diffusion equation suffers an intractable behavior near the singularity.

Although the LAGM developed in this paper has no extraordinary device at all for stability like up-winding FDM or SUPG, the numerical solutions are accurate enough to maintain the second order accurate numerical solutions. The LAGM has several major advantages in computation. First, we use the local axial Green's function corresponding to the approximated convection coefficient on a local axial line rather than using the exact form of axial Green's function. This makes the method efficient because it enables us to find the axial Green's function in a tractable form that can drastically reduce the computational cost. Second, compared to other methods like FDM or FEM, it is easier to construct the local axial lines in an arbitrary domain on which the numerical solutions are calculated without loss of accuracy. The surprising fact is that randomly distributed axial lines are even available in our method.

In addition to these advantages, it should be emphasized that the stiffness matrix resulting from the formulation in terms of our LAGM is perfectly sparse ($O(M)$). In contrast, the sparsity in the original AGM depends on the complexity of the domain and $O(M^{\frac{3}{2}})$ in 2D if M is the number of cross points between axial lines, like nodes in FDM.

2. Localized axial Green's function method: LAGM

2.1. Decomposing the convection–diffusion equation into one-dimensional problems

Assume Ω is an arbitrary open bounded domain in $\mathbb{R}^2$. We consider the convection–diffusion equation in Ω,

$$-\nabla\cdot(\epsilon\nabla u) + \mathbf{U}(\mathbf{x})\cdot\nabla u = f, \quad \text{in } \Omega, \tag{1}$$

$$u = u^{\partial\Omega}, \quad \text{on } \partial\Omega, \tag{2}$$

where we assume that $\epsilon(\mathbf{x}) > 0$ is the diffusion coefficient continuous in general, and $\mathbf{U}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}))$ the convection coefficient bounded in magnitude. If the above equation is dimensionless and ϵ is constant, then the Péclet number is defined as the reciprocal of ϵ. The high Péclet number implies the small diffusion compared to the convection, or the convection-dominated case.

To figure out the LAGM for Eqs. (1) and (2), at the first stage, we split the partial differential equation in (1) into axially decomposed equations in the following form

$$-(\epsilon u_x)_x + a_1 u_x = \frac{f}{2} + \phi, \quad \text{in } \Omega, \tag{3}$$

$$-(\epsilon u_y)_y + a_2 u_y = \frac{f}{2} - \phi, \quad \text{in } \Omega, \tag{4}$$