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Short note

Spectrally accurate numerical solution of hypersingular boundary integral equations for three-dimensional electromagnetic wave scattering problems

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In this paper we extend the spectrally accurate algorithms developed by Ganesh et al. in [2,3] to the numerical solution of a modified combined-field integral equation (M-CFIE) for electromagnetic wave scattering in three dimensions.

1. Introduction

We are concerned with the numerical solution of the scattering problem of time-harmonic electromagnetic waves by a three-dimensional perfectly conducting obstacle. We assume that the perfect conductor is represented by a bounded domain Ω in $\mathbb{R}^3$ with a simply connected closed boundary Γ of class $\mathscr{C}^1$ at least. Let Ω^c denote the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$ and $\boldsymbol{n}$ denote the outer unit normal vector to the boundary Γ. Let κ denote the exterior wavenumber. The propagation of electromagnetic waves is governed by the system of Maxwell equations and the time-harmonic Maxwell system can be reduced to the second order equation for the electric field only. In this case the scattering problem is formulated as follows [1,6]: Given an incident electric wave $\boldsymbol{E}^{inc}$ which is assumed to solve the homogeneous second order Maxwell equation $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}^{inc} - \kappa^2\boldsymbol{E}^{inc} = 0$ in a neighborhood of the boundary Γ, find the electric scattered wave $\boldsymbol{E}^s$ solution to the Maxwell equation $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}^s - \kappa^2\boldsymbol{E}^s = 0$ in Ω^c and satisfying the boundary condition $\boldsymbol{n}\times(\boldsymbol{E}^s + \boldsymbol{E}^{inc}) = \mathbf{0}$ on Γ. In addition the scattered field $\boldsymbol{E}^s$ has to satisfy the Silver–Müller radiation condition: $|\,\mathbf{curl}\,\boldsymbol{E}^s(\mathbf{x})\times\mathbf{x} - i\kappa|\mathbf{x}|\boldsymbol{E}^s(\mathbf{x})| \overset{|\mathbf{x}|\to+\infty}{\longrightarrow} 0$. This scattering problem can be reduced in several different ways to a uniquely solvable modified combined-field boundary integral equation (M-CFIE) for all positive real values of the exterior wave number [1,7]. In Section 2 we review the standard indirect approaches.

Efficient numerical solution of such scattering problems is of practical interest for various industrial applications, especially in the area of inverse obstacle scattering. Among all the existing numerical methods to solve integral equations, we focus here on spectral methods. Ganesh and Hawkins already proposed in [3], and references therein, several spectrally accurate methods to implement the magnetic field integral equation (MFIE). It consists in transporting the MFIE on the unit sphere $\mathbb{S}^2$ via a normal transformation acting from the tangent plane to the boundary Γ onto the tangent plane to the

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unit sphere, so that one only has to seek a solution in terms of tangential vector spherical harmonics. To treat numerically the M-CFIE, we need to implement the hypersingular part of the electromagnetic single layer boundary integral operator. This requires a transformation that moves the nullspace of the surface divergence on Γ onto the nullspace of the surface divergence on $\mathbb{S}^2$. The implementation of a hypersingular integral can then be avoided by involving integration by parts and surface derivatives of the vector spherical harmonics that can be computed analytically. Such a transformation is well known as the Piola transform. In Section 3 we give the reformulation of the integral equation in spherical coordinates, based on this approach. We explain in Section 4 how this equation can be implemented by using a combination of the high order spectral algorithm developed in [2] for integral equations arising in acoustic scattering and the one developed in [3] for the MFIE. Numerical results in the resonance frequency region for the unit sphere and in the medium frequency region for a variety of three-dimensional convex and non-convex smooth obstacles are presented in Section 5 to show the efficiency of the method. We obtain convergence rates similar to those reported in Ganesh et al. [2,3].

2. The solution of the perfect conductor problem

We denote by $H^s_{loc}(\overline{\Omega^c})$ and $H^s(\Gamma)$ the standard (local in the case of the exterior domain) complex valued, Hilbertian Sobolev space of order $s \in \mathbb{R}$ defined on $\overline{\Omega^c}$ and Γ respectively (with the convention $H^0 = L^2$). Spaces of vector functions will be denoted by boldface letters, thus $\boldsymbol{H}^s = (H^s)^3$. We use the tangential gradient denoted by ∇_Γ, the tangential vector curl denoted by $\mathbf{curl}_\Gamma$, the surface divergence denoted by $\operatorname{div}_\Gamma$ and the surface scalar curl denoted by $\operatorname{curl}_\Gamma$. For their definitions we refer to [1,4,6] and references therein. For $s \in \mathbb{R}$, we introduce the Hilbert space $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma) = \{\boldsymbol{j} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma);\ \boldsymbol{j}\cdot\boldsymbol{n} = 0 \text{ and } \operatorname{div}_\Gamma \boldsymbol{j} \in H^{-\frac{1}{2}}(\Gamma)\}$, endowed with the norm $\|\cdot\|_{\boldsymbol{H}^s_{\operatorname{div}}(\Gamma)} = (\|\cdot\|^2_{\boldsymbol{H}^s(\Gamma)} + \|\operatorname{div}_\Gamma \cdot\|^2_{H^s(\Gamma)})^{1/2}$. We set $\boldsymbol{H}_{loc}(\mathbf{curl}, \overline{\Omega^c}) = \{\boldsymbol{v} \in \boldsymbol{L}^2_{loc}(\overline{\Omega^c}) : \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2_{loc}(\overline{\Omega^c})\}$ and we define $\boldsymbol{H}_{loc}(\mathbf{curl}\,\mathbf{curl}, \overline{\Omega^c})$ in the same way. Recall that for a vector function $\boldsymbol{u} \in \boldsymbol{H}_{loc}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}_{loc}(\mathbf{curl}\,\mathbf{curl}, \Omega)$, the traces $\boldsymbol{n} \times \boldsymbol{u}_{|\Gamma}$ and $\boldsymbol{n} \times \mathbf{curl}\,\boldsymbol{u}_{|\Gamma}$ are in $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$. The dual space of $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$ for the $\boldsymbol{L}^2$ duality product is $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{curl}}(\Gamma)$ and the exterior product with the normal vector defines a bicontinuous isomorphism between $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$ and $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{curl}}(\Gamma)$.

For any $\kappa > 0$, let $\Phi(\kappa, \mathbf{z}) = \frac{e^{i\kappa|\mathbf{z}|}}{4\pi|\mathbf{z}|}$ be the fundamental solution of the Helmholtz equation $\Delta u + \kappa^2 u = 0$. Here we consider the modified combined-field integral equation (M-CFIE) method based on the layer ansatz [1, Section 6.4]:

$$\boldsymbol{E}^s(\mathbf{x}) = \int_\Gamma \mathbf{curl}^{\mathbf{x}}\big\{\Phi(\kappa, \mathbf{x}-\mathbf{y})\boldsymbol{j}(\mathbf{y})\big\}ds(\mathbf{y}) + i\eta \int_\Gamma \frac{1}{\kappa}\mathbf{curl}\,\mathbf{curl}^{\mathbf{x}}\big\{\Phi(\kappa, \mathbf{x}-\mathbf{y})\boldsymbol{\Lambda}\boldsymbol{j}(\mathbf{y})\big\}ds(\mathbf{y}), \tag{2.1}$$

where $\boldsymbol{\Lambda}$ is a bounded operator from $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$ to itself, self-adjoint and elliptic for the bilinear form

$$(\boldsymbol{j}, \boldsymbol{m}) \mapsto \int_\Gamma \boldsymbol{j}\cdot(\boldsymbol{n}\times\boldsymbol{m})\,ds \tag{2.2}$$

and η is a non vanishing real constant.

We assume $\boldsymbol{f} = -\boldsymbol{n} \times \boldsymbol{E}^{inc} \in \boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$. By the jump relations, the field $\boldsymbol{E}^s \in \boldsymbol{H}_{loc}(\mathbf{curl}, \overline{\Omega^c}) \cap \boldsymbol{H}_{loc}(\mathbf{curl}\,\mathbf{curl}, \overline{\Omega^c})$ given by (2.1) solves the perfect conductor problem if the density $\boldsymbol{j}$ solves the following integral equation [1, Eq. (6.49)]

$$(\mathrm{I} + M_\kappa + i\eta C_\kappa \boldsymbol{\Lambda})\boldsymbol{j} = 2\boldsymbol{f} \quad \text{on } \Gamma. \tag{2.3}$$

Here the single layer potential C_κ and the double layer potential M_κ are defined by [1, Eqs. (6.30) and (6.40)]

$$M_\kappa \boldsymbol{j}(\mathbf{x}) = \int_\Gamma \boldsymbol{n}(\mathbf{x}) \times \mathbf{curl}^{\mathbf{x}}\big\{2\Phi(\kappa, \mathbf{x}-\mathbf{y})\boldsymbol{j}(\mathbf{y})\big\}ds(\mathbf{y}),$$

$$\begin{aligned} C_\kappa \boldsymbol{j}(\mathbf{x}) &= \frac{1}{\kappa}\int_\Gamma \boldsymbol{n}(\mathbf{x}) \times \mathbf{curl}\,\mathbf{curl}^{\mathbf{x}}\big\{2\Phi(\kappa, \mathbf{x}-\mathbf{y})\boldsymbol{j}(\mathbf{y})\big\}ds(\mathbf{y}) \\ &= \kappa\boldsymbol{n}(\mathbf{x}) \times \int_\Gamma 2\Phi(\kappa, \mathbf{x}-\mathbf{y})\boldsymbol{j}(\mathbf{y})ds(\mathbf{y}) - \frac{1}{\kappa}\mathbf{curl}_\Gamma \int_\Gamma 2\Phi(\kappa, \mathbf{x}-\mathbf{y})\operatorname{div}_\Gamma \boldsymbol{j}(\mathbf{y})ds(\mathbf{y}). \end{aligned}$$

The operator $M_\kappa : \boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma) \to \boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$ is compact and the operator C_κ has a hypersingular kernel but is bounded on $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$. The operator $\boldsymbol{\Lambda}$ is then chosen such that $(\mathrm{I} + M_\kappa + i\eta C_\kappa\boldsymbol{\Lambda})$ is a Fredholm operator of index zero. Kress first proposed in [1] a compact regularization $\boldsymbol{\Lambda}\boldsymbol{j} = \boldsymbol{n} \times S_0^2\boldsymbol{j}$ where S_0 is the single layer boundary integral operator of the Laplace equation, thus $(\mathrm{I} + M_\kappa + i\eta C_\kappa\boldsymbol{\Lambda})$ is a Fredholm operator of the second kind. One can also use the elliptic and invertible operator which is a variant of the operator C_κ [7] defined on $\boldsymbol{H}^{-\frac{1}{2}}_{\operatorname{div}}(\Gamma)$ by $\boldsymbol{\Lambda}\boldsymbol{j} = \boldsymbol{n} \times S_0\boldsymbol{j} + \mathbf{curl}_\Gamma S_0 \operatorname{div}_\Gamma \boldsymbol{j}$.