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Multiwavelet troubled-cell indicator for discontinuity detection of discontinuous Galerkin schemes

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ABSTRACT

In this paper, we introduce a new *global* troubled-cell indicator for the discontinuous Galerkin (DG) method in one- and two-dimensions. This is done by taking advantage of the global expression of the DG method and re-expanding it in terms of a multiwavelet basis, which is a sum of the global average and finer details on different levels. Examining the higher level difference coefficients acts as a troubled-cell indicator, thus avoiding unnecessary increased computational cost of a new expansion. In two-dimensions the multiwavelet decomposition uses combinations of scaling functions and multiwavelets in the x- and y-directions for improved troubled-cell indication. By using such a troubled-cell indicator, we are able to reduce the computational cost by avoiding limiting in smooth regions. We present numerical examples in one- and two-dimensions and compare our troubled-cell indicator to the subcell resolution technique of Harten (1989) [11] and the shock detector of Krivodonova et al. (2004) [18], which were previously investigated by Qiu and Shu (2005) [23].

1. Introduction and motivation

Nonlinear hyperbolic partial differential equations are often solved using the Runge–Kutta discontinuous Galerkin (DG) method [7,6,5,8]. In practical applications, initial conditions may contain discontinuities, or the solution of a nonlinear equation may develop a shock at a certain time. To efficiently apply DG in the case of discontinuous solutions, limiting techniques are used to reduce the spurious oscillations that develop in discontinuous regions. Examples of these limiters are the minmod-based TVB limiter [7], TVD limiters [4], WENO [25,26], and the moment limiter [17]. Unfortunately, most of the limiters do not work well for higher order approximations (they limit smooth extrema), or multidimensional cases. In order to limit the correct elements, a troubled-cell indicator can be used. This procedure detects discontinuous regions, where the use of a limiter is necessary. A limiter is then applied only in the identified troubled cells. In general, this leads to more accurate results in smooth regions, and reduces the computational cost significantly.

There are a variety of troubled-cell indicators, some that are tied to the limiting procedure and others that are separate from this procedure. A few of the important methods of troubled-cell indication are minmod [7], Harten's subcell resolution [11], moment limiters [17], monotonicity preserving limiters [29], and the shock detector of Krivodonova et al. (KXRCF) [18]. These methods for indicating troubled cells were explored and compared by Qiu et al. in [23]. They did this in

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order to improve the performance of a WENO-based limiter for DG. They found that there is no universally better performing method for every problem. However, they did find that the minmod based limiter with a suitably chosen parameter, Harten's method and the KXRCF shock detector performed better than other methods.

In this paper, we introduce a new troubled-cell indicator using ideas from a multiwavelet formulation. We explain the relation between the multiwavelet expansion and the DG formulation, [2,3]. This multiwavelet expansion is decomposed into a sum of a global average and finer details on different levels. The absolute averages of the highest decomposition level act as a troubled-cell indicator, which suddenly increase in the neighborhood of a discontinuity [21]. In two-dimensions, the multiwavelet decomposition uses combinations of scaling functions and multiwavelets in the x-, and y-direction. This is the reason why we are able to detect the exact locations of discontinuities in the x-, or y-direction, or in one of the diagonal directions [21].

This multiwavelet troubled-cell indicator takes a different tack than most troubled-cell indicators. Instead of only considering local information, this technique uses *global* information to detect the troubled cells. This technique performs well, even in the vicinity of a strong shock with weaker local shocks such as the double Mach reflection problem. It was recently pointed out by Zaide et al. [32] that for systems, using local information one will find three different shock locations in each of the conserved variables. However, by using global information, we obtain one location for the shock(s). This allows us to implement a limiter in a smaller region, thereby reducing the time for computation. We demonstrate the robust performance of our indicator on a variety of test problems, using the moment limiter in the identified troubled cells [17]. The results using our new troubled-cell indicator will be compared with the method of Harten and the KXRCF shock detector and we show that it reliably performs better and more efficiently.

The outline of this paper is as follows: in Section 2 we present the relevant background information in discontinuous Galerkin methods and multiwavelets. In Section 3 we introduce our new global multiwavelet troubled-cell indicator. The effectivity of this new method compared with existing methods is presented in Section 4 for standard numerical examples. We conclude with a discussion of our method and future work in Section 5.

2. Background

In this section, relevant background information regarding discontinuous Galerkin methods [4,22], multiwavelets [2], troubled-cell indicators [23,18] and limiting [17] is presented. This information will be used in Section 3 to develop our new troubled-cell indicator. To begin, an explanation of the discontinuous Galerkin method in two dimensions is given.

2.1. The discontinuous Galerkin method

In order to describe the discontinuous Galerkin (DG) method, consider the following partial differential equation on a rectangular domain $\Omega \in \mathbb{R}^2$:

$$u_t + \nabla \cdot \mathbf{f}(u) = 0, \quad \mathbf{x} = (x, y) \in \Omega, \ t \geqslant 0; \tag{2.1a}$$

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.1b}$$

where $u = u(\mathbf{x}, t)$, and $\mathbf{f}(u) = (f(u), g(u))^\top$ is the flux function.

To discretize in space, Ω is divided into $(N_x + 1) \times (N_y + 1)$ rectangular elements given by

$$I_{ij} = \left\{(x, y){:}\ x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],\ y \in (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\right\}, \quad i = 0, \ldots, N_x,\ j = 0, \ldots, N_y.$$

On each element, the chosen approximation space is defined as

$$V_h(I_{ij}) = \left\{v \in \mathbb{Q}^k(I_{ij})\right\}, \quad i = 0, \ldots, N_x,\ j = 0, \ldots, N_y.$$

Here, $\mathbb{Q}^k$ is the space of polynomials, $\mathbb{Q}^k = \operatorname{span}\{x^m y^n{:}\ 0 \leqslant m, n \leqslant k\}$.

For simplicity, the basis of $\mathbb{Q}^k$ is constructed using a tensor product of the scaled Legendre polynomials, $\phi_{\ell_x}(x)\phi_{\ell_y}(y)$, ℓ_x, $\ell_y \in \{0, \ldots, k\}$. These functions are defined as

$$\phi_\ell(x) = \sqrt{\ell + \frac{1}{2}} P^{(\ell)}(x), \tag{2.2}$$

where $P^{(\ell)}$ is the Legendre polynomial of degree $\ell \in \mathbb{N}$. We note that these functions are pairwise orthonormal:

$$\langle \phi_\ell, \phi_m \rangle = \int_{-1}^{1} \phi_\ell(x)\phi_m(x)\, dx = \delta_{\ell m}. \tag{2.3}$$

This choice of basis functions provides ease when pairing the discontinuous Galerkin method with a multiwavelet approximation.