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Local discontinuous Galerkin numerical solutions of non-Newtonian incompressible flows modeled by p-Navier–Stokes equations

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ABSTRACT

We present a Local Discontinuous Galerkin (LDG) method for solving non-Newtonian incompressible flow problems. The problems are modeled by p-type Navier–Stokes equations, the extra stress tensor follows a p-power rule (p-NS). The aim of this paper is to present an efficient way for discretizing the governing equations by an LDG method, choosing both equal and mixed order local polynomial space for velocity and pressure. The velocity gradient is introduced as an auxiliary variable and the p-NS system is decomposed into a first order system including the projection of the nonlinear stress tensor components to the local discontinuous space. Every equation resulting *from the splitting* of the extra stress tensor is discretized under the DG element by element technique. In the divergence constraint equation, an artificial compressibility term (time derivative of the pressure with a small parameter) is added and the first order terms are expressed in a divergence form, and are discretized by utilizing the Lax–Friedrichs numerical fluxes. An upwind wave analysis is applied on the outflow boundary for constructing artificial outflow boundary conditions. For the time discretization, s-stage Diagonally Implicit Runge–Kutta schemes are applied. Numerical experiments on problems with known exact solutions are performed for verifying the expected convergence rates of the method. Benchmark problems are considered in order to check the performance of the method for solving flow problems described by p-NS systems.

1. Introduction

In many chemical engineering and bio-mechanical flow applications non-Newtonian fluid models are applied in order to describe the physical problem, [1,2]. In these models, the viscosity appears to be a nonlinear function of the shear rate. The wide range of use of these models has led to an effort for developing high-order accuracy numerical methods for discretizing the resulting system and simulating the flow problems, [3,4].

Depending on the characteristics of the flow, different mathematical viscous stress splitting forms have been proposed and finite element/volume methods (FE/FV) with stabilization techniques (e.g. stream line upwind Petrov–Galerkin FE, discontinuous Galerkin methodology) have been developed for obtaining high-order accurate solutions, [5–9].

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Also, in the literature, non-Newtonian incompressible fluid flow problems in *primitive variable formulation*, have been studied [10–12]. In [13], the main features of LDG methods are analyzed for *quasi-Newtonian Stokes fluid problems*, problems were the viscosity is C^1 function of the shear rate.

In this work, we study the case where the viscosity and the shear rate follow a *p-power rule*, and the extra stress tensor has the so-called *p-structure*, [14]. Hereafter, we will denote this Navier–Stokes system by the abbreviation p-NS. Local Discontinuous Galerkin methods for p-NS cannot be found in the literature, and the main goal of this paper is to address this issue. The method proposed here for discretizing the extra stress tensor is an extension of the LDG method presented in [15,16] for p-type nonlinear elliptic systems. The numerical fluxes are designed to be compatible with the p-structure of the stress tensor ensuring the stability of the whole LDG method.

On the other hand, during the last two decades, there has been an increasing interest on devising DG and LDG methods for the numerical solution for Navier–Stokes equations of incompressible problems ($p = 2$), see a review in [17,18] and a theoretical analysis for mixed order space methods in [19,20]. Cockburn et al. in [21], based on [19,20] introduced certain pressure stabilization terms in the discrete analogue of the divergence constraint, which make possible the application of the same k-polynomial order space for velocity and pressure. In addition, a post-processing procedure is applied (projection of the discrete solution to BDMK spaces) and finally globally divergence free velocity solution is obtained. Lastly, hybridizable DG methods (HYDG), [22], have been proposed for steady and time-dependent Navier–Stokes equations in [23].

The LDG discretization presented here for p-NS, contains the following the main steps. The extra stress tensor is expressed in divergence form by introducing the $\mathbf{L} = \nabla\mathbf{u}$ as auxiliary variable. The divergence constraint is relaxed by adding an artificial compressibility (pressure time derivative) term with a small parameter, and thus the remainder first order terms of the original system are gathered together in one flux and are discretized by applying hyperbolic problem DG methodology. The expression of the first order terms in a single flux, makes possible an upwind wave analysis in order to construct artificial outflow boundary conditions, since *traction boundary conditions* for p-NS, have not been proposed so far for numerical applications on realistic flow problems. On the other hand, we mention that the *do nothing* boundary conditions usually applied for Newtonian incompressible flows, are not appropriate for non-Newtonian flow problems, [24]. In all numerical examples we consider regular conforming triangulations, with mixed local polynomial order $\mathbb{P}^2 - \mathbb{P}^1$ ($\mathbb{P}^2$ for velocity – $\mathbb{P}^1$ for pressure) and equal order, $\mathbb{P}^1 - \mathbb{P}^1$ and $\mathbb{P}^2 - \mathbb{P}^2$.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem. In Section 3 we analyze the LDG numerical scheme, where we describe the discretization of the viscous and the first order terms separately, also the definition of the corresponding numerical fluxes is given. In the last paragraph of the section, we present the artificial outflow boundary conditions. In Section 4, we present the implicit s-stage Runge–Kutta scheme for the time discretization. Finally, in Section 5, numerical examples for certificating the expected convergence rate of the scheme and validating the behavior of the method for approximating solutions of p-NS are shown. The paper closes with the conclusions.

2. The model problem

Consider a viscous non-Newtonian incompressible fluid flow in a bounded domain $\Omega \subset \mathbb{R}^d$ (for the problems here $d = 2$) with smooth boundary $\Gamma = \partial\Omega$. The fluid velocity $\mathbf{u} = (u_1, \dots, u_d)$ and the pressure P, are governed by the p-Navier–Stokes equations (momentum and continuity) of incompressible flow

$$\mathbf{u}_t - \operatorname{div}\mathbf{S}(\mathbf{Du}) + (\nabla\mathbf{u})\mathbf{u} + \nabla P = \mathbf{f}, \quad \text{in } \Omega, \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{1b}$$

where t is the time, $\mathbf{S}(\mathbf{Du})$ denotes the extra stress tensor which depends on the symmetric velocity gradient $\mathbf{Du} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)$ and $\mathbf{f} = (f_1, \dots, f_d)$ is the external body force.

The relation of $\mathbf{S}$ and $\mathbf{Du}$ is nonlinear and is expressed by a p-power law, [14],

$$\mathbf{S}(\mathbf{Du}) = \frac{1}{Re}\big(\delta + |\mathbf{Du}|\big)^{p-2}\mathbf{Du}, \tag{2}$$

where $|\mathbf{Du}| = \sqrt{\sum_{(i,j)=1}^{d}(Du)_{ij}^2}$ is the shear rate measure, $\delta \geqslant 0$ is a constant, Re is the Reynolds number, and $p > 1$. As can be seen from (2), the non-Newtonian fluid motions differ by the choice of the exponent p. For $p = 2$, system (1) describes Newtonian fluid motions (constant viscosity, classical incompressible Navier–Stokes system). For $p < 2$ the fluid exhibits shear-thinning behavior (viscosity decreasing by increasing the shear rate measure) and for $p > 2$ shear-thickening behavior.

The system (1) is completed by the initial and boundary conditions on $\partial\Omega$. We split $\partial\Omega$ in to three disjoint subsets, the inflow part Γ_{infl}, the solid wall Γ_w and the outflow part Γ_{outf}, where usually the boundary conditions are prescribed as follows, [25,26],

$$\mathbf{u} = \mathbf{u}_{infl}, \quad \text{on } \Gamma_{infl}, \tag{3a}$$

$$\big(\mathbf{S}(\mathbf{Du}) - PI\big) \cdot \mathbf{n}_{\Gamma_{outf}} = \mathbf{q}, \quad \text{on } \Gamma_{outf}, \tag{3b}$$

$$\mathbf{u} = \mathbf{u}_w(=0). \quad \text{on } \Gamma_w, \tag{3c}$$