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A sixth order hybrid finite difference scheme based on the minimized dispersion and controllable dissipation technique

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A B S T R A C T

The dispersion and dissipation properties of a scheme are of great importance for the simulation of flow fields which involve a broad range of length scales. In order to improve the spectral properties of the finite difference scheme, the authors have previously proposed the idea of optimizing the dispersion and dissipation properties separately and a fourth order scheme based on the minimized dispersion and controllable dissipation (MDCD) technique is thus constructed [29]. In the present paper, we further investigate this technique and extend it to a sixth order finite difference scheme to solve the Euler and Navier–Stokes equations. The dispersion properties of the scheme is firstly optimized by minimizing an elaborately designed integrated error function. Then the dispersion–dissipation condition which is newly derived by Hu and Adams [30] is introduced to supply sufficient dissipation to damp the unresolved wavenumbers. Furthermore, the optimized scheme is blended with an optimized Weighted Essentially Non-Oscillation (WENO) scheme to make it possible for the discontinuity-capturing. In this process, the approximation–dispersion–relation (ADR) approach is employed to optimize the spectral properties of the nonlinear scheme to yield the true wave propagation behavior of the finite difference scheme. Several benchmark test problems, which include broadband fluctuations and strong shock waves, are solved to validate the high-resolution, the good discontinuity-capturing capability and the high-efficiency of the proposed scheme.

1. Introduction

For the high-fidelity simulation of compressible flows with a broadband of length scales and discontinuities, the numerical schemes must be capable of resolving these scales and at the same time handling the flow discontinuities. On one hand, the numerical scheme must be of low-dissipation and low-dispersion to accurately capture the small scales both in amplitude and phase. On the other hand, spurious oscillations may be produced in the vicinity of the discontinuities if the scheme cannot supply sufficient dissipations. The paradoxical nature of these requirements presents great challenge for the design of such kind of numerical schemes. For example, the well-established spectral scheme [1–3] and compact scheme [4–6] have excellent spectral properties. However, they inevitable produce the numerical oscillations near the shock regions and may cause failure of the simulation. Contrarily, the widely used shock-capturing schemes such as the Essentially

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Non-Oscillatory (ENO) [7] and Weighted Essentially Non-Oscillatory (WENO) [8] can properly capture the shock waves without spurious oscillations, but they may be too dissipative for the detailed simulations of problems related to turbulence and computational aero-acoustics [9] due to their large dissipation.

One promising choice to overcome these drawbacks is to combine the shock-capturing schemes with schemes which have good spectral properties. In this line of thought, Adams and Shariff [10] proposed a hybrid scheme in which the non-conservative form of the upwind compact scheme was used in the shock-free regions while the conservative ENO scheme was employed near the discontinuities. Pirozzoli [11] devised a more accurate hybrid compact-WENO scheme to solve the Euler equations by replacing the ENO scheme with WENO scheme and the non-conservative compact scheme with the conservative compact scheme. This work was further improved by Ren et al. [12], who designed a weighted function to avoid the abrupt transition between the compact scheme and the WENO scheme. Hill and Pullin [13] combined the tuned central difference scheme with a WENO scheme to form a hybrid scheme and used it in the large eddy simulations (LES) of compressible flows with strong shocks. Other contributors in this aspects include Shen and Yang [14], who developed hybrid finite compact-WENO schemes and Costa and Don [15], who developed hybrid central-WENO scheme and hybrid spectral-WENO scheme. The most important issue of the hybrid scheme is the design of the shock sensor or discontinuities detector in order to switch or blend between the sub-schemes and there have been lots of discussions on this open problem [12,16–18].

Another logical approach can be built by optimizing the spectral properties of the shock-capturing scheme. There are two aspects that should be concerned in the optimization procedure. One aspect is to optimize the spectral properties of the scheme at the expense of the formal order of accuracy. The formal order accuracy of a numerical scheme supplies information on the asymptotic convergence rate to the exact solution with the refinement of the grid points [19]. However, it does not provide the actual error of a scheme on a given mesh. It has been validated that the dispersion-optimized schemes may resolve the high frequency components of the solutions better than some traditional finite difference schemes with higher order of accuracy. Thus, such spectral-optimized schemes are suitable for the simulation of problem with a wide range of length scales, such as turbulence, aero-acoustics, etc. Lele [4] used the Fourier analysis to measure the phase error of a scheme and developed a family of compact schemes with spectral-like resolution. Tam and Webb [20] proposed the idea of optimizing the coefficients of the explicit finite difference scheme to satisfactorily resolve the short wavelength structures. These schemes were denoted as dispersion–relation–preserving (DRP) schemes and were further developed by many researchers. Wang and Chen [21] optimized the optimal weights of the WENO scheme in the wavenumber space under the guidance of DRP. Martín et al. [9] optimized the candidate stencils of the WENO scheme by minimizing a delicately designed integrated error function. Moreover, they added an additional candidate stencil to make the stencils symmetric rather than upwind-biased in the process of reconstruction. Another aspect is to improve the nonlinear mechanism of the shock-capturing schemes to optimize their spectral properties. Take the widely-used WENO scheme as an example, lots of works have been devoted to analyze and improve the nonlinear adaptation weights of it. Henrick et al. [22] proposed the WENO-M scheme in which the nonlinear weights are pushed to the optimal values by a regularization process. The WENO-Z scheme developed by Borges et al. [23,24] normalizes the smoothness indicators by a reference value which can drive the nonlinear weights to their optimal counterpart more quickly than the original WENO scheme. Taylor et al. [25] devised the WENO-RL scheme by directly switching the nonlinear weights to the optimal weights when the relative ratio between the smoothness indicator is small. This approach clings the dispersion curve together with the exact line up to a certain range of wavenumbers and thus can enhance the capability of the scheme in resolving the elements with medium wavenumbers. More recently, Hu and Adams [26] developed an adaptive central-upwind 6th-order WENO (WENO-CU6) scheme which adapts between central and upwind schemes smoothly by introducing a new reference smoothness indicator. They further incorporated a physically-motivated scale formulation to the WENO-CU6 scheme which leads to a scale separation of contributions from resolved scales and unresolved scales [27]. Numerical examples imply this scheme leads to a physically consistent implicit subgrid-scale (SGS) for the turbulence simulation while maintaining good shock-capturing capabilities. Arshed et al. [28] elaborately analyzed errors from linear and nonlinear weights of the WENO scheme and minimized them for broadband applications with shock waves.

Concerning the spectral optimization of a scheme, it is generally accepted the dispersion error should be minimized according to some chosen criteria. However, there are no general guidelines on how the dissipation should be optimized in the literature. On one hand, a scheme with very large dissipation is obviously not suitable for detailed simulation of turbulent flows. On the other hand, the minimum dissipation produced by the central scheme is not sufficient to suppress the numerical oscillations. Pirozzoli [11] has pointed out that a certain amount of dissipation is necessary to damp the high wavenumber elements as they are incorrect both in phase and amplitude. However, he did not give the necessary dissipation in quantity. Sun et al. [29] found that the dissipation and dispersion of a class of explicit finite difference schemes can be controlled independently. They presented a 4th-order finite difference scheme with Minimized Dispersion and Controllable Dissipation (MDCD) properties. The dispersion properties are optimized by the minimization of an integrated error function while the dissipation is controlled by variation of one free parameter. The main advantage of the MDCD scheme is that they can change the dissipation of the resulting scheme without corrupting the already optimized dispersion properties. However, they still supply no criterion for the selection of the dissipation and the dissipation have to be chosen manually by numerical tests. To cope with this difficulty, Hu et al. [30] mentioned there should be a compromise between the dispersion and dissipation properties. They derived the dispersion–dissipation condition for the explicit finite difference schemes which