Journal of Computational Physics 270 (2014) 378–401

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Fast Huygens sweeping methods for Helmholtz equations in inhomogeneous media in the high frequency regime

Songting Luo [a], Jianliang Qian [b,*], Robert Burridge [c]

[a] Department of Mathematics, Iowa State University, Ames, IA 50011, USA
[b] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
[c] Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA

ARTICLE INFO

Article history:
Received 26 November 2013
Received in revised form 20 February 2014
Accepted 31 March 2014
Available online 8 April 2014

Keywords:
Eikonal
Helmholtz
High-frequency wave
Huygens sweeping

ABSTRACT

In some applications, it is reasonable to assume that geodesics (rays) have a consistent orientation so that the Helmholtz equation may be viewed as an evolution equation in one of the spatial directions. With such applications in mind, we propose a new Eulerian computational geometrical-optics method, dubbed the fast Huygens sweeping method, for computing Green functions of Helmholtz equations in inhomogeneous media in the high-frequency regime and in the presence of caustics. The first novelty of the new method is that the Huygens–Kirchhoff secondary source principle is used to integrate many locally valid asymptotic solutions to yield a globally valid asymptotic solution so that caustics associated with the usual geometrical-optics ansatz can be treated automatically. The second novelty is that a butterfly algorithm is adapted to carry out the matrix–vector products induced by the Huygens–Kirchhoff integration in $O(N\log N)$ operations, where N is the total number of mesh points, and the proportionality constant depends on the desired accuracy and is independent of the frequency parameter. To reduce the storage of the resulting traveltime and amplitude tables, we compress each table into a linear combination of tensor-product based multivariate Chebyshev polynomials so that the information of each table is encoded into a small number of Chebyshev coefficients.

The new method enjoys the following desired features: (1) it precomputes a set of local traveltime and amplitude tables; (2) it automatically takes care of caustics; (3) it constructs Green functions of the Helmholtz equation for arbitrary frequencies and for many point sources; (4) for a specified number of points per wavelength it constructs each Green function in nearly optimal complexity in terms of the total number of mesh points, where the prefactor of the complexity only depends on the specified accuracy and is independent of the frequency parameter.

Both two-dimensional (2-D) and three-dimensional (3-D) numerical experiments are presented to demonstrate the performance and accuracy of the new method.

* Corresponding author.
E-mail addresses: luos@iastate.edu (S. Luo), qian@math.msu.edu (J. Qian), burridge137@gmail.com (R. Burridge).

http://dx.doi.org/10.1016/j.jcp.2014.03.066
0021-9991/

1. Introduction

We consider the Helmholtz equation with a point-source condition,

$$\nabla_{\mathbf{r}}^2 U + \frac{\omega^2}{v^2} U = -\delta(\mathbf{r} - \mathbf{r}_0), \tag{1}$$

where the Sommerfeld radiation condition is imposed at infinity, $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the source point, $U(\mathbf{r}, \omega; \mathbf{r}_0)$ is the wave field, ω is the frequency, $v(\mathbf{r})$ is the wave speed, and $\nabla_{\mathbf{r}}^2$ denotes the Laplacian at $\mathbf{r} = (x, y, z)$. This equation arises in a variety of physical applications, ranging from acoustics, elasticity, electromagnetics to geophysics. Therefore, it is highly desirable to develop efficient and accurate numerical methods for this equation. Because the solution of the Helmholtz equation represents the spatial factor of a time-harmonic solution of the time-dependent wave equation, it is highly oscillatory when the frequency parameter ω is large. Since it is very costly for a direct method to resolve these oscillations, asymptotic methods such as geometrical optics are sought to deal with such difficulties. In this paper, we propose a new Eulerian computational geometrical-optics method, which we call the fast Huygens sweeping method, for the Helmholtz equations in inhomogeneous media in the high-frequency regime and in the presence of caustics. The new method is based on the Huygens secondary source principle and the recent development of butterfly algorithms for constructing low-rank matrix approximations.

To motivate the new method, we apply the geometrical-optics large-ω ansatz to the Helmholtz equation, yielding the eikonal equation for traveltime and the transport equation for amplitude, respectively. These two equations are weakly coupled in the sense that the eikonal equation needs to be solved first to provide necessary coefficients for the transport equation for the amplitude. Because the eikonal equation is a first-order nonlinear partial differential equation (PDE), in general it does not have a globally defined smooth solution. The concept of viscosity solution was invented to single out a uniquely defined weak solution among many possible generalized solutions for such nonlinear first-order PDEs, and the resulting viscosity solution for the eikonal equation is the so-called first-arrival traveltime [26], which is continuous everywhere but not necessarily differentiable everywhere. The difficulty arises exactly when the viscosity solution fails to be differentiable, and the resulting gradient of the traveltime function is discontinuous. Consequently, the linear transport equation for the squared amplitude in the conservation form has discontinuous coefficients; theoretically, such linear transport equations with discontinuous coefficients are not well understood. Computationally, nevertheless, we may design high-order numerical methods to solve the eikonal and transport equations in physical space to obtain the traveltime and amplitude functions; the resulting two functions can be substituted into the geometrical-optics ansatz to obtain an "asymptotic" solution for the Helmholtz equation. But when we compare this asymptotic solution with the direct solution of the Helmholtz equation, we find that these two solutions are different "globally". Mathematically, this difference may be traced back to the multivaluedness of the traveltime function and related caustics [5], which can be detected by applying the method of characteristics to the eikonal equation with the appropriate point-source condition. One of the major observations in [29,31] is that the large-ω ansatz yields faithful asymptotic solutions to the Helmholtz equation before caustics occur; in other words, this ansatz can be used *locally*. Then the natural question is: how to obtain a globally defined, faithful asymptotic solution to the Helmholtz equation by using this large-ω ansatz locally? The answer is provided by the Huygens secondary source principle, which roughly states that the wave field from a source can be replaced by the field radiated by equivalent secondary sources on a surface which encloses the source.

It has been shown in [35,46] that the traveltime function for the eikonal equation with a point-source condition is locally smooth in the neighborhood of the source except at the source point itself; this implies that caustics will not develop right away on the expanding wavefront away from the source. Therefore, in a local neighborhood of the point source, the traveltime and amplitude functions resulting from solving the eikonal and transport equations are smooth except at the point source, and they yield a valid asymptotic Green function in that local neighborhood.

To go beyond caustics, we will make some assumptions for the Helmholtz equation under consideration. For some seismic applications, it is natural to assume that geodesics (rays) have a consistent orientation (such as directed downwards) so that the Helmholtz equation can be viewed as an evolution equation in one of the spatial directions. Consequently, we first partition the computational domain into several subdomains which enclose the primary point source, and we make use of the Huygens principle by setting up secondary sources on the closed boundaries of those subdomains. More importantly, the partitioning should satisfy the requirement that the large-ω ansatz for each secondary source on the subdomain boundaries generates an asymptotic solution which is valid in the corresponding subdomain, and this requirement can be achieved by computing the traveltime function for that secondary source point in a small neighborhood where the resulting traveltime field is smooth. According to this construction each secondary source provides a locally valid Green function. To synthesize information from those secondary sources, we use the Huygens–Kirchhoff integral identity to carry out an integration along enclosed boundaries so that we can compute the global Green function for the primary source at any observation point inside the subdomain under consideration. In this way we sweep through the whole domain to obtain the global Green function for the primary source, and caustics are implicitly and automatically taken care of during this sweeping process.

The issue is now concentrated on how to implement the above sweeping strategy efficiently. To tackle this challenging problem, we must surmount several obstacles. The first obstacle is that the traveltime and amplitude functions for the eikonal and transport equations with point-source conditions have upwind singularities at the source point, making it difficult to compute these two functions with high-order accuracy; moreover, the occurrence of the Laplacian of the traveltime