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# A frame-invariant vector limiter for flux corrected nodal remap in arbitrary Lagrangian–Eulerian flow computations

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ABSTRACT

This article describes a frame-invariant vector limiter for Flux-Corrected Transport (FCT) numerical methods. Our approach relies on an objective vector projection, and, because of its intrinsic structure, the proposed approach can be generalized with ease to higher-order tensor fields.

The proposed concept is applied to nodal finite element formulations and the so-called algebraic FCT paradigm, but the ideas pursued here are very general and also apply to more general instantiations of flux-corrected transport.

Specifically, we consider the arbitrary Lagrangian–Eulerian (ALE) equations of compressible inviscid flows. In addition to the geometric conservation law (GCL) and the local extreme diminishing (LED) property of the original scalar limiters, the proposed approach ensures frame invariance (objectivity) for vectors. Particularly, we use an ALE strategy based on a two-stage, Lagrangian plus mesh remap (data transfer based on conservative interpolation), in which remap and limiting are performed in a synchronized way. The proposed approach is however of general applicability, is not limited to a specific ALE implementation, and can easily be generalized to computations with standard (monolithic) ALE or Eulerian reference frames.

The significance of the frame-invariant limiter for vectors is demonstrated in computations of compressible materials under extreme load conditions. Extensive testing in two and three dimensions demonstrates that the proposed limiter greatly enhances the robustness and reliability of the existing methods under typical computational scenarios.



## 1. Introduction

The present article describes a frame-invariant limiting strategy for vectors in arbitrary Lagrangian–Eulerian (ALE) computations with nodal finite elements. Here, the frame invariance property or *objectivity* means that the formulation of the method should remain unchanged under any arbitrary (possibly time-dependent) rigid transformation given by rotations and translations (see for more details [1–3]).

In [4–6], frame invariance was found to be very important in designing stabilized methods for ALE computations. This issue, however, has been long overlooked in the context of flux limiting, a procedure designed to enhance the nonlinear stability of computational fluid dynamics solvers, without precluding their overall accuracy. A recent work in this direction is by Luttwak and Falcovitz [7–9] in the context of slope limiters for finite volume methods. The authors would also like to

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point to recent work by Maire [10], in which it is recognized the importance of ensuring frame invariance in the construction of finite volume methods for Lagrangian flow simulations, and the very recent work by Velechovský et al. [11] and Kuchařík et al. [12], mostly aimed at defining limiting strategies that preserve certain flow symmetries (e.g., radial symmetry).

The present work, on the other hand, aims at designing frame-invariant limiters within the context of nodal finite element methods for advection systems in general and remap algorithms in ALE computations in particular. Here remap refers to a grid data transfer method based on conservative interpolation.

Algebraic FCT methods [13] were designed for nodal finite element computations, and were derived from the original FCT algorithms [14,15]. These methods were originally designed for scalar problems in multiple dimensions using general (unstructured) grids. Hence, for the sake of simplicity and without any ambiguity, algebraic FCT methods will be referred to in what follows as FCT methods. We note that limiting for vector quantities remains an open question in the FCT community.

Common practice is to apply the FCT limiters only to scalars like density and energy (or pressure) [16], or apply them to each component of a vector (for example, flow velocity or linear momentum) [13]. Both approaches have their drawbacks, since, as shown in Section 4, applying limiters only to scalar variables like density and internal energy is not sufficient to avoid spurious oscillations in kinetic and internal energy, and component-wise limiting of vector variables causes strong mesh dependences in the overall solution. Specifically, these oscillations tend to occur near contact discontinuities across which the magnitude of the tangential velocity has a large jump, or in high vorticity regions of the flow. These issues are documented here in the context of ALE–Remap algorithms, but it is conceivable that also more standard ALE or Eulerian algorithms suffer from the same problems.

These observations strongly suggest that limiters applied to vector fields (such as the flow velocity) should be considered to enhance the robustness of the flow solver, but that component-by-component application of limiters is undesirable in that it destroys the frame invariance properties of the overall method.

In this work, we develop a simple frame-invariant limiting strategy for vectors using nodal finite elements. The methodology is an extension of scalar limiting strategies. In particular, we first identify a vector field, termed as the projection direction, to estimate at each node the direction along which the oscillations most likely occur. Next, we project the vector field to be limited (specifically, velocity or momentum) along this projection direction. Finally, an FCT limiter analogous to the one in [17] is applied to compute nodal limiter coefficients for this scalar field. We conclude by synchronizing the limiters for all fields of interest in the computations. By design, this approach is provably frame-invariant as long as the nodal projection direction constitutes a vector field (i.e., a frame-invariant quantity).

The effectiveness of our vector limiting strategy relies on the particular projection direction used. We propose two strategies to compute the projection direction at each node, by either using the flow density gradient or the unit eigenvector associated with the principal eigenvalue of the deviator of the symmetric gradient of the velocity (or any vector to be limited). The former is especially effective for shock hydrodynamics because the density gradient typically points in the direction normal to shock fronts or contact discontinuities. The latter is of more general applicability, since it does not require any information other than the vector itself and the mesh position.

The numerical performance of both approaches are demonstrated by extensive testing in two- and three-dimensional examples in the context of Lagrangian/ALE shock hydrodynamics, and compared against scalar and component-wise limiters. To put into context these computational studies, the adopted flow solver is an ALE algorithm, composed of three stages, following an operator splitting procedure: A pure Lagrangian computation is first performed, then the nodes of the Lagrangian mesh are repositioned to improve the overall grid quality and finally the numerical solution is transferred from the old to the new grid. This last step is usually referred to as "remap" in Lagrangian/ALE shock hydrodynamics. More specifically, the remap stage consists in transferring the numerical solution from a given mesh to a perturbed mesh with the same connectivity but different node locations (cf. ALE algorithms based on reconnection [18] or edge-swapping [19], which both change the mesh connectivity). At each time step, the remap procedure can be interpreted as an advection problem, solved with an algebraic flux-corrected transport (FCT) method [13,20–25]. Our approach incorporates the geometric conservation law (GCL) [26] and inherits conservation and local extremum diminishing (LED) [27] properties that are typical of FCT schemes.

We point out that the algebraic FCT strategy used in this work is not the only possible option, and alternative strategies for unstructured FCT computations on nodal (dual volume) discretizations can be used in the present context.

The remainder of the paper is organized as follows: Section 2 briefly reviews the arbitrary Lagrangian–Eulerian remap using nodal finite elements and the FCT limiters for scalar fields during this procedure; the frame-invariant property of the whole procedure is studied in Section 3, which also contains the two new invariant strategies, and a mathematical proof of invariance (objectivity). The numerical performances of the proposed vector limiters are compared to scalar and component-wise limiters by solving two- and three-dimensional examples in Section 4. Conclusions are summarized in Section 5.

## 2. General concepts in arbitrary Lagrangian–Eulerian remap

This section reviews the principles of arbitrary Lagrangian–Eulerian remap, the finite element analysis of the remap problem using piecewise linear elements, and algebraic flux-correction with Zalesak limiter for remap of scalar fields. More details can be found in previous work [17]: Only the key concepts and equations necessary for the frame-invariance analysis conducted in the next section are presented here.