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High-order conservative reconstruction schemes for finite volume methods in cylindrical and spherical coordinates

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ABSTRACT

High-order reconstruction schemes for the solution of hyperbolic conservation laws in orthogonal curvilinear coordinates are revised in the finite volume approach. The formulation employs a piecewise polynomial approximation to the zone-average values to reconstruct left and right interface states from within a computational zone to arbitrary order of accuracy by inverting a Vandermonde-like linear system of equations with spatially varying coefficients. The approach is general and can be used on uniform and non-uniform meshes although explicit expressions are derived for polynomials from second to fifth degree in cylindrical and spherical geometries with uniform grid spacing. It is shown that, in regions of large curvature, the resulting expressions differ considerably from their Cartesian counterparts and that the lack of such corrections can severely degrade the accuracy of the solution close to the coordinate origin. Limiting techniques and monotonicity constraints are revised for conventional reconstruction schemes, namely, the piecewise linear method (PLM), third-order weighted essentially non-oscillatory (WENO) scheme and the piecewise parabolic method (PPM).

The performance of the improved reconstruction schemes is investigated in a number of selected numerical benchmarks involving the solution of both scalar and systems of nonlinear equations (such as the equations of gas dynamics and magnetohydrodynamics) in cylindrical and spherical geometries in one and two dimensions. Results confirm that the proposed approach yields considerably smaller errors, higher convergence rates and it avoid spurious numerical effects at a symmetry axis.

1. Introduction

Unsteady, time-dependent compressible flows often involve complex flow interactions featuring both continuous and discontinuous waves. Numerical computations based on finite-volume (FV) discretizations have now established as a reliable tool to model such flows and delivering oscillation-free stable solutions while preserving conservation of relevant physical quantities such as mass, momentum and energy. FV methods (for a review see the books from [1,2]) rely on a conservative discretization based on the integral formulation of the underlying system of partial differential equations (PDEs) where volume averages (rather than point values) are evolved in time. Average quantities can thus vary only when an unbalance exists between the fluxes entering and leaving the region boundary. The computation of the interface flux is the heart of these methods and it is usually achieved by employing proper upwinding techniques that rely on the solution of a Riemann problem between discontinuous left and right states at cell interfaces. These states are reconstructed from the volume averages of the solution and several techniques are available in literature, e.g., second-order TVD methods [3–5,1,2], third-order piecewise parabolic method [6], essentially non-oscillatory (ENO [7]) and weighted essentially non-oscillatory (WENO [8,9], see also [10–13] and references therein), monotonicity preserving (MP [14]) schemes. Generally speaking, the

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reconstruction is a two-step process where a high-order accurate estimate of the interface values is first provided and later modified (or limited) to fulfill monotonicity constraints.

Traditionally, most reconstruction techniques have been devised for Cartesian geometry and a vast literature exists on this subject. Curvilinear systems, nevertheless, are often preferred and employed in modeling many scientific applications such as, for instance, geophysical or atmospheric flows, flows in turbomachinery, astrophysical accretion disks orbiting around a central object or, more simply, flows with rotational symmetry around a vertical axis.

In this respect, it should be stressed that little attention has been devoted to the development of high-order finite volume methods in curvilinear coordinate systems [15–19] and that straightforward application of Cartesian-based reconstruction schemes to a curvilinear grid may suffer from a number of drawbacks and inconsistencies that have often been overlooked. For second-order accurate schemes, this has already been demonstrated by a number of authors (e.g., [15,16] and, more recently, [19]) who recognized that volume averages should be assigned to the centroid of volume rather than the geometrical cell center. Higher than second-order schemes, on the other hand, still deserve a more careful treatment since simple-minded extensions of plane-parallel reconstruction methods may easily lead to incorrect results and severely compromise the accuracy of the solution in proximity of a symmetry axis. In the original formulation of the Piecewise Parabolic Method (PPM [6]), for instance, the authors suggested to perform the reconstruction in the volume coordinate (rather than the linear one) so that the same algorithm used for a Cartesian mesh could be employed on a cylindrical or spherical radial grid. In doing so, however, the resulting interface states become formally first-order accurate even for smooth flows. This shortcoming was addressed in [17] (see also [18]) and corrected by first interpolating the indefinite integral of a conserved fluid quantity and then differentiating the resulting polynomial with respect to the linear coordinate to obtain the desired point values at a cell interface. Although formally correct, this approach has the disadvantage of being potentially singular at the coordinate origin and that the resulting interface states may loose one or even two orders of accuracy.

The intent of the present work is to re-formulate and improve, in the context of orthogonal curvilinear coordinates, some of the most widely used reconstruction techniques employed by FV methods. The proposed formulation is based on a piecewise polynomial reconstruction from the volume averages of conserved quantities lying in adjacent zones yielding, in one dimension, interface states that are formally correct to arbitrary order of accuracy. This approach is presented in Section 2 for a scalar conservation law and it is general enough to be employed on regularly – as well as irregularly – spaced grids. Closed form solutions are derived on uniform radial grids in cylindrical and spherical geometries for polynomials of second up to fifth degree. For a more complex coordinate system and/or non-uniform grids the reconstruction process can still be carried out by using numerical quadrature and/or the solution of a linear system of equations at the beginning of the computation. In order to suppress spurious oscillations, conventional limiting techniques for second-order TVD, WENO and PPM schemes are revised in Section 3 for the case of cylindrical and spherical geometries. Extension to nonlinear systems of equations is treated in Section 4 where reconstruction from primitive variables and integration of geometrical source terms are discussed. Finally, in Section 5, the proposed schemes are tested and compared on one- and two-dimensional selected test problems in cylindrical and spherical geometries. Both scalar hyperbolic conservation laws and nonlinear systems of equation are considered.

2. Problem formulation

2.1. Finite volume discretization in curvilinear coordinates

Given an orthogonal system of coordinates (x_1, x_2, x_3) with unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ and scale factors (h_1, h_2, h_3), we now wish to solve the scalar conservation law

$$\frac{\partial Q}{\partial t} + \nabla\cdot\mathbf{F} = S, \tag{1}$$

where Q is a conserved fluid quantity, $\mathbf{F} = (F_1, F_2, F_3)$ is the corresponding flux vector, S is a source term while the divergence operator takes the form

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1) + \frac{\partial}{\partial x_2}(h_1h_3F_2) + \frac{\partial}{\partial x_3}(h_1h_2F_3)\right]. \tag{2}$$

Eq. (1) is discretized on a computational domain divided into $N_1 \times N_2 \times N_3$ cells (or zones) with lower and upper coordinate bounds respectively given by $(x_{1,i-\frac12}, x_{2,j-\frac12}, x_{3,k-\frac12})$ and $(x_{1,i+\frac12}, x_{2,j+\frac12}, x_{3,k+\frac12})$ so that the mesh spacings are denoted with

$$\Delta x_{1,i} = x_{1,i+\frac12} - x_{1,i-\frac12}; \qquad \Delta x_{2,j} = x_{2,j+\frac12} - x_{2,j-\frac12}; \qquad \Delta x_{3,k} = x_{3,k+\frac12} - x_{3,k-\frac12}, \tag{3}$$

while the cell volume is defined by

$$\Delta\mathcal{V}_{i,j,k} = \int_{x_{3,k-\frac12}}^{x_{3,k+\frac12}}\int_{x_{2,j-\frac12}}^{x_{2,j+\frac12}}\int_{x_{1,i-\frac12}}^{x_{1,i+\frac12}} h_1h_2h_3\, dx_1dx_2dx_3. \tag{4}$$