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A second order PVM flux limiter method. Application to magnetohydrodynamics and shallow stratified flows

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A B S T R A C T

In this work we propose a second order flux limiter finite volume method, named PVM-2U-FL, that only uses information of the two external waves of the hyperbolic system. This method could be seen as a natural extension of the well known WAF method introduced by E.F. Toro in [23]. We prove that independently of the number of unknowns of the 1D system, it recovers the second order accuracy at regular zones, while in presence of discontinuities, the scheme degenerates to PVM-2U method, which can be seen as an improvement of the HLL method (see [6,10]). Another interesting property of the method is that it does not need any spectral decomposition of the Jacobian or Roe matrix associated to the flux function. Therefore, it can be easily applied to systems with a large number of unknowns or in situations where no analytical expression of the eigenvalues or eigenvectors are known. In this work, we apply the proposed method to magnetohydrodynamics and to stratified multilayer flows. Comparison with the two-waves WAF and HLL-MUSCL methods are also presented. The numerical results show that PVM-2U-FL is the most efficient and accurate among them.

1. Introduction

The goal of this article is to design a robust, simple and fast second order flux limiter numerical scheme to solve one-dimensional hyperbolic systems. An interesting technique to obtain second order accurate and robust schemes is to use a non-linear combination of first and second order methods in terms of flux limiters functions. An example of this type of scheme can be defined by a suitable combination of Roe method (which is only first order) near discontinuities, and the Lax–Wendroff method (which is second order in space and time) in regular areas. Note that the previous scheme requires the explicit knowledge of the eigenstructure of the system, which is not straightforward for some hyperbolic systems, making this scheme computationally expensive in those cases.

It is also well known that the use of incomplete Riemann solvers as Rusanov, Lax–Friedrichs, HLL, among others (see [13,27,7,10,31]) allows one to reduce the computing time required by a Roe solver (see, for instance, [11], [16] and [20]). Although Roe scheme gives, in general, a better resolution of the discontinuities than incomplete Riemann solvers, it may be indistinguishable when combined with high order methods.

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In [6] Castro and Fernández-Nieto introduce a family of incomplete simple Riemann solvers named as PVM (Polynomial Viscosity Matrix), for conservative and nonconservative hyperbolic systems, defined in terms of viscosity matrices computed by a suitable polynomial evaluation of a Roe linearization, that overcome the difficulty of the computation of the spectral decomposition of Roe matrices. PVM schemes can be seen as the natural extension of the one proposed in [10] for balance laws, and, more generally, for nonconservative systems.

An interesting numerical scheme that uses flux limiters functions is the WAF (Weighted Average Flux) method, introduced by Toro in [23]. It is a one-step Godunov-type method to solve hyperbolic conservation laws that achieves second order accuracy by averaging the solution of the Riemann problem with piecewise constant initial data. As it is well known, due to Godunov's theorem, linear schemes with high order accuracy generate spurious oscillations near discontinuities. To avoid this problem, WAF method uses flux limiter functions. The resulting scheme is a non-linear TVD (Total Variation Diminishing) scheme with second order accuracy. WAF scheme has been extensively used to approximate hyperbolic systems, see for example [24,25,12,19,26,30]. It has been also used as the base of higher order numerical solvers (see [29]).

Nevertheless, WAF method needs the explicit knowledge of the structure of the approximated Riemann problem to achieve second order accuracy. For example, if we only consider the information of the two external waves and we use the HLL intermediate flux we obtain a WAF method – that we will name in what follows HLL-WAF method – that has second order accuracy for 1D 2×2 hyperbolic conservative systems.

The main objective of this paper is to obtain a new flux limiter scheme that only uses the information of the two external waves, like the HLL-WAF scheme, and that achieves second order accuracy for 1D $N \times N$ hyperbolic systems with $N \geqslant 2$. The resulting scheme can be seen as a natural extension of the original HLL-WAF scheme and it is defined in terms of a non-linear combination of a suitable PVM scheme, that is first order, with the second order Lax–Wendroff scheme.

The paper is organized as follows: in Section 2, first we summarize how WAF and, in particular, HLL-WAF methods are derived. Next, HLL-WAF method is rewritten as a non-linear combination of two PVM schemes. In Section 3, the new flux limiter scheme is defined and finally, some numerical tests for the ideal magnetohydrodynamics and the multilayer shallow-water systems are presented. Comparison with HLL-WAF and the second order HLL methods with MUSCL (see [14,15,32] and [35]) are also presented.

2. Preliminaries

In this section we summarize the derivation of WAF method introduced by E.F. Toro in [23]. Let us consider the conservative hyperbolic system

$$w_t + F(w)_x = 0, \quad x \in [0, L],\ t \in [0, T], \tag{2.1}$$

where $w(x,t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$ and F is a regular function from $\mathcal{O}$ to $\mathbb{R}^N$.

Let us consider a partition of the domain $\{x_i\}_i = \{i\Delta x\}_i$ where, by simplicity, Δx is supposed to be constant, and we denote $t^n = n\Delta t$ where Δt is the time step. A finite volume method in conservative form to approximate (2.1) can be written as

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right), \tag{2.2}$$

where w_i^n denotes an approximation of the mean value of the solution on the control volume $(x_{i-1/2}, x_{i+1/2})$ at time $t = t^n$:

$$w_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w\left(x, t^n\right) dx,$$

and $\mathcal{F}_{i+1/2}^n = \mathcal{F}(w_i^n, w_{i+1}^n)$ denotes the numerical flux function that characterizes each method.

Let us consider a Riemann problem associated to (2.1) with initial data w_i^n and w_{i+1}^n:

$$\begin{cases} w_t + F(w)_x = 0, \\ w(x,0) = \begin{cases} w_i & x < 0; \\ w_{i+1} & x > 0, \end{cases} \end{cases} \tag{2.3}$$

where we have removed superindex n for sake of simplicity. In what follows, the dependency of the intercell $i + 1/2$ will be dropped for clarity if there is no ambiguity.

Le us denote S_l, $l = 1, \ldots, N$, the approximation of the characteristic velocities and let us consider the computational cell $V = [-\Delta x/2, \Delta x/2] \times [0, \Delta t]$. Then, WAF numerical flux is obtained by integrating the physical flux in V using the midpoint rule for the time integral:

$$\mathcal{F}_{i+1/2}^{\mathrm{WAF}} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} F\left(\tilde{w}\left(x, \frac{\Delta t}{2}\right)\right) dx, \tag{2.4}$$

where $\tilde{w}$ is an approximated solution of the Riemann problem (2.3), composed by $N + 1$ constant states.