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Dispersive behaviour of high order finite element schemes for the one-way wave equation

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ABSTRACT

We study the ability of high order numerical methods to propagate discrete waves at the same speed as the physical waves in the case of the one-way wave equation. A detailed analysis of the finite element method is presented including an explicit form for the discrete dispersion relation and a complete characterisation of the numerical Bloch waves admitted by the scheme. A comparison is made with the spectral element method and the discontinuous Galerkin method with centred fluxes. It is shown that all schemes admit a spurious mode. The spectral element method is always inferior to the finite element and discontinuous Galerkin schemes; a somewhat surprising result in view of the fact that, in the case of the *second order wave equation*, the spectral element method propagates waves with an accuracy *superior* to that of the finite element scheme. The comparative behaviour of the finite element and discontinuous Galerkin scheme is also somewhat surprising: the accuracy of the finite element method is superior to that of the discontinuous Galerkin method in the case of elements of odd order by two orders of accuracy, but worse, again by two orders of accuracy, in the case of elements of even order.

1. Introduction and summary of main results

Consider the one-way wave equation for a given wave-speed $c > 0$,

$$\partial_t u + c\partial_x u = 0, \quad x \in \mathbb{R},\ t > 0 \tag{1}$$

with suitable initial data. A key feature of the equation is the existence of non-trivial, spatially propagating solutions for each given temporal frequency ω,

$$u(x,t) = e^{i\omega t} U(x) \tag{2}$$

where $U(x) = e^{-ikx}$, $k = \omega/c$. The relation between the wavenumber and the temporal frequency is known as the *dispersion relation* for the continuous problem. The function U satisfies a *Bloch wave condition*

$$U(x+h) = \lambda U(x), \quad x \in \mathbb{R},\ h \in \mathbb{R} \tag{3}$$

where $\lambda = e^{-ikh}$ is the Floquet multiplier.

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Let $X_{h,N}$ denote the space of continuous, piecewise polynomials of degree N on the grid $h\mathbb{Z}$,

$$X_{h,N} = \left\{ v \in C(\mathbb{R}){:}\ v\,|_{(x_m,x_{m+1})} \in \mathbb{P}_N,\ m \in \mathbb{Z} \right\} \tag{4}$$

where $x_m = mh$. A semi-discrete approximation of the one-way wave equation may be defined by seeking $u_{h,N} \in X_{h,N}$ such that

$$\int_{\mathbb{R}} (\partial_t u_{h,N} + c\partial_x u_{h,N})\bar{v}\, \mathrm{d}x = 0, \quad v \in X_{h,N} \tag{5}$$

along with appropriate initial conditions. A key issue [4] when assessing any spatial discretisation scheme for the one-way wave equation is the existence of non-trivial Bloch wave solutions of the discrete problem (5). These solutions again take the form (2),

$$u_{h,N}(x,t) = e^{i\omega t} U_{h,N}(x) \tag{6}$$

with the essential difference that the function $U_{h,N}$ must belong to the discrete space $X_{h,N}$, and satisfy a discrete Bloch wave condition

$$U_{h,N}(x+h) = \lambda_{h,N} U_{h,N}(x), \quad x \in \mathbb{R} \tag{7}$$

with the discrete Floquet multiplier $\lambda_{h,N}$ depending on the mesh-size h and the polynomial degree N.

The ability of the numerical scheme to propagate waves in space faithful to the true propagating waves depends on the accuracy with which the discrete Floquet multiplier approximates the true Floquet multiplier. The relative accuracy $R_{h,N}$ of the approximation is defined by

$$R_{h,N} = \frac{\lambda - \lambda_{h,N}}{\lambda} \tag{8}$$

and our aim is to study the behaviour of this ratio as $\omega h/c \to 0$, for any polynomial order N. Some authors prefer to introduce a *discrete wavenumber*, $k_{h,N}$, satisfying

$$e^{-ihk_{h,N}} = \lambda_{h,N} \tag{9}$$

and to study the relative accuracy of the approximation $k \approx k_{h,N}$, where $k = \omega/c$ is the true wavenumber, given by

$$E_{h,N} = \frac{k - k_{h,N}}{k}. \tag{10}$$

These measures are related in the case where $k - k_{h,N}$ is small as follows

$$R_{h,N} = \frac{e^{-ikh} - e^{-ik_{h,N}h}}{e^{ikh}} = 1 - e^{-i(k_{h,N}-k)h} \approx -i(k - k_{h,N}) = -ikE_{h,N}. \tag{11}$$

As such, the choice of whether to study $R_{h,N}$ or $E_{h,N}$ is purely a matter of taste in the case where $k - k_{h,N}$ is small. However, it should be borne in mind that the condition (9) does not define a unique value of $k_{h,N}$. Care must be taken in selecting the value of $k_{h,N}$ satisfying (9) appropriately in order to avoid drawing incorrect conclusions. Moreover, if $k - k_{h,N}$ is *not* small, then there is no simple relation between $R_{h,N}$ and $E_{h,N}$. For these reasons, our preference is to study the relative accuracy of the discrete Floquet multiplier directly, since it is this quantity that appears in the Bloch wave condition and is uniquely defined.

Our first result establishes an algebraic condition on the discrete Floquet multiplier in terms of the order N, the mesh-size h and the wavenumber ω/c under which a non-trivial discrete Bloch wave may exist.

Theorem 1. *There exists a non-trivial Bloch wave solution of problem* (5) *of the form*

$$u_{h,N}(x,t) = e^{i\omega t} \sum_{m\in\mathbb{Z}} \lambda_{h,N}^m \phi(x - mh) \tag{12}$$

where $\phi \in X_{h,N}$, *if and only if* $\lambda_{h,N}$ *is a solution of the algebraic equation*

$$\overline{v_N(\omega h/c)}\big(\lambda - q_N(\omega h/c)\big) + (-1)^N v_N(\omega h/c)\left(\frac{1}{\lambda} - \overline{q_N(\omega h/c)}\right) = 0, \tag{13}$$

where $q_N(\Omega) = w_N(\Omega)/\overline{v_N(\Omega)}$, $v_N(\Omega)$ *and* $w_N(\Omega)$ *are defined in Theorem 4.*

The function ϕ appearing in the Bloch wave expansion (12) is a piecewise polynomial supported on $(-h,h)$ that depends on the polynomial degree N and $\Omega = \omega h/c$. The function is constructed as part of the proof of Theorem 1 given in Section 3. Fig. 1 shows the function in the case $h = 1$ and $\omega = 2c$ for polynomial degree N from 1 to 6.