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Numerical solution of distributed order fractional differential equations

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ABSTRACT

In this paper a method for the numerical solution of distributed order FDEs (fractional differential equations) of a general form is presented. The method applies to both linear and nonlinear equations. The Caputo type fractional derivative is employed. The distributed order FDE is approximated with a multi-term FDE, which is then solved by adjusting appropriately the numerical method developed for multi-term FDEs by Katsikadelis. Several example equations are solved and the response of mechanical systems described by such equations is studied. The convergence and the accuracy of the method for linear and nonlinear equations are demonstrated through well corroborated numerical results.

1. Introduction

Fractional differential equations (FDEs) appear more and more frequently in different research areas and engineering applications. Various physical phenomena in the fields of viscoelasticity, diffusion procedures, relaxation vibrations, electrochemistry, etc. are successfully described by ordinary or partial differential equations involving derivatives of fractional (non-integer) order [1,2]. The related literature on fractional differential equations is rich [3–5]. More recently, several problems in mathematical physics and engineering have been modeled via distributed order FDEs [6–12]. An interesting application field of these equations appears in viscoelasticity. Atanackovic [13] has generalized the multi-term fractional derivative viscoelastic model to a derivative model of distributed order by replacing the finite sums with integrals in the domain of orders. Based on this model Atanackovic and his co-workers have analyzed the response of several systems such as the fractional distributed order oscillator [14] and the distributed order fractional wave equation [15–17]. These problems lead to distributed order FDEs, which the authors treated by analytical methods. As the realm of the distributed order FDEs describing the real life response of physical systems grows, the demand for numerical solution to treat these equations becomes more pronounced in order to overcome the mathematical complexity of analytical solutions. Therefore, the development of effective and easy-to-use numerical schemes for solving such equations acquires an increasing interest. The numerical analysis of a certain form of distributed order FDEs with a numerical scheme to obtain numerical results was presented in [18]. The development, however, for efficient numerical methods to solve linear and nonlinear distributed order FDEs is still an important issue.

In this paper a numerical method is presented for solving distributed order FDEs of the form

$$\int_{\alpha}^{\beta} F\left[p, D_c^p u(t)\right] dp + G\left[t, u(t), D_c^{\alpha_i} u(t)\right] = f(t) \tag{1}$$

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where α and β are positive real numbers; D_c^p is the Caputo type fractional derivative of order p defined as

$$D_c^p u(t) = \begin{cases} \frac{1}{\Gamma(m-p)} \int_0^t \frac{u^{(m)}(\tau)}{(t-\tau)^{p+1-m}} d\tau, & m-1 < p < m, \\ u^{(m)}(t), & p = m \end{cases} \tag{2}$$

with m being an integer; α_i ($\alpha_1 < \alpha_2 < \cdots < a_r$) positive real numbers and F, G linear or nonlinear functions. Since the Caputo derivative is employed, initial conditions with physical significance can be specified, namely $u^{(k)}(0) = u_0^{(k)}$, $k = 0, 1, \ldots, l-1$, where $l = \max\{\mathrm{ceil}(b), \mathrm{ceil}(\alpha_r)\}$. The basic ingredients of the method consist in:

(a) Approximating the integral with a finite sum using a simple quadrature rule. Thus the distributed order FDE is converted into a multi-term FDE; and
(b) Developing a numerical method to solve the resulting multi-term FDE.

In this analysis the trapezoidal rule is employed to approximate the integral and the numerical procedure in [19] is applied to solve the resulting multi-term FDE after discretization of the integration interval. The form of the treated distributed order FDEs is more general than in [18], while the lower bound of the integration interval $[\alpha, \beta]$ is not necessarily zero.

The presented method inherits all merits of the scheme presented in [19] regarding the accuracy, stability and convergence. It is simple to implement. There is no computational complexity in constructing the solution algorithms, while they are reasonably easy to program. Several example equations are solved. The convergence and the accuracy of the method are demonstrated through well corroborated numerical examples. Nevertheless, an analytical investigation of the estimation of the convergence would validate the findings. This might be a subject of further work. The efficiency of the scheme is also illustrated by solving the distributed order FDEs describing the dynamic response of mechanical systems. The method can be further developed to solve partial distributed FDEs in bounded domains following the ideas presented in [20].

2. Linear equations

We shall illustrate the method by considering first linear equations of the form

$$\int_\alpha^\beta \phi(p) D_c^p u(t)\, dp = f(t). \tag{3}$$

Without restricting the generality, the solution procedure is presented for $0 \leqslant \alpha < \beta \leqslant 2$.

The initial conditions depend on $\mathrm{ceil}(\beta)$. Thus we have

$$u(0) = u_0 \quad \text{if } 0 < \beta \leqslant 1, \tag{4a}$$

or

$$u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0 \quad \text{if } 1 < \beta \leqslant 2. \tag{4b}$$

We distinguish the following two cases: (i) $\alpha = 0$, (ii) $0 < \alpha$.

2.1. Integration interval $[0, \beta]$

Approximating the integral in Eq. (3) with a sum using the trapezoidal rule with K equal intervals we obtain

$$\Delta p \left[\frac{\phi_0}{2} D_c^{p_0} u + \phi_1 D_c^{p_1} u + \phi_2 D_c^{p_2} u + \cdots + \phi_{K-1} D_c^{p_{K-1}} u + \frac{\phi_K}{2} D_c^{p_K} u \right] = f(t), \quad \Delta p = \beta / K \tag{5}$$

with $D_c^{p_0} u = u$ and $D_c^{p_K} u = D_c^\beta u$.

2.1.1. The AEM solution and the numerical scheme

The analog equation method (AEM) described in [19] is employed to solve the multi-term fractional differential equation (5) subject to the initial conditions (4a), (4b). This method is presented below as applied to the problem at hand.

Let $u = u(t)$ be the sought solution. Then, if the operator D_c^β is applied to u we have

$$D_c^\beta u = q(t), \quad t > 0 \tag{6}$$

where $q(t)$ is a fictitious source, unknown in the first instance. Eq. (6) is the analog equation of (5). It indicates that the solution of Eq. (5) can be obtained by solving Eq. (6) with the initial conditions (4a), (4b). This is accomplished as follows.