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On the accuracy and efficiency of discontinuous Galerkin, spectral difference and correction procedure via reconstruction methods

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ABSTRACT

Numerical accuracy and efficiency of several discontinuous high-order methods, including the quadrature-based discontinuous Galerkin (QDG), nodal discontinuous Galerkin (NDG), spectral difference (SD) and flux reconstruction/correction procedure via reconstruction (FR/CPR), for the conservation laws are analyzed and compared on both linear and curved quadrilateral elements. On linear elements, all the above schemes are one-dimensional in each natural coordinate direction. However, on curved elements, not all schemes can be reduced to a one-dimensional form, although the SD and CPR formulations remain one-dimensional by design. The efficiency and accuracy of various formulations are compared on highly skewed curved elements. Several benchmark problems are simulated to further evaluate the performance of these schemes.

1. Introduction

Adaptive high-order (order of accuracy $\geqslant 3$) methods have received considerable interest in the computational fluid dynamics (CFD) community recently due to their potential of delivering higher accuracy with lower computational cost than the low-order methods for problems involving complex physics and geometry, such as aero-acoustic noise prediction and vortex dominated flows. Readers are referred to several books [1–3] and reviews on these methods [4,5].

The most popular high-order method for compressible flow simulations is arguably the discontinuous Galerkin (DG) method. It was first introduced by Reed and Hill [6] for neutron transport equations in 1973 and then developed by Cockburn and Shu [7,8] and Cockburn et al. [9] for hyperbolic conservation laws. Bassi and Rebay [10,11] applied the DG discretization to the Euler and Navier–Stokes equations, and emphasized the importance of the proper treatment of the curved boundaries. Some comprehensive reviews on the DG development for both hyperbolic and elliptic problems can be found in [12,13]. Depending on how the degrees-of-freedom (DOFs) are chosen, various DG implementations have different numerical properties and efficiencies. In the most straightforward implementation of the DG approach, the Gauss quadrature is used to compute the surface and volume integrals in the weak formulation. We will call this DG implementation quadrature-based DG or QDG. A more efficient implementation, the nodal DG (NDG), was developed by Hesthaven and Warburton [2]. The NDG approach avoids the use of quadrature by expanding the (nonlinear) flux on the same Lagrange basis employed by the solution. This approach was also explored in the quadrature-free implementation of DG by Atkins and Shu [14]. Generally speaking, QDG is almost always more expensive than NDG due to the numerical quadrature and extra interpolation. For linear conservation laws, QDG and NDG are identical in accuracy. For nonlinear conservation laws, QDG is

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more accurate as NDG may run into problems associated with aliasing errors. Note that the DG methods on quadrilateral elements are also referred as DG spectral element methods [15–18].

In order to avoid the explicit numerical integrals, some finite-difference-like discontinuous formulations have been developed. One successful high-order approach of this kind is the staggered-grid (SG) multi-domain spectral method [19] or the spectral difference (SD) method [20,21]. This method directly reconstructs a flux polynomial based on fluxes on a given nodal set called flux points. Then the derivatives of the flux polynomial are used to update the solutions at the solution points. On quadrilateral or hexahedral elements, the SD method is one-dimensional by design, even for high-order elements. This is significant, as the computational cost is much less than that of the DG approaches in multiple dimensions. Later, it was found that the computational cost of SD can be further decreased as Huynh [22] and Van den Abeele et al. [23] confirmed that the SD method only depends on the location of the flux points, and the staggered-grid configuration, i.e. staggered distribution of solution points and flux points, is not necessary. This suggests that computation efficiency can be further improved if the solution points coincide with the flux points [23].

In 2007, Huynh [22] introduced the flux reconstruction (FR) approach for one-dimensional conservation law. The FR approach shares with SD the philosophy of solving the differential form of the equation. The key difference, however, is that FR allows the flux polynomial to be reconstructed by a procedure significantly more general than the interpolation procedure used by SD. Depending on how the flux polynomial is defined, the FR approach unifies many existing methods such as the DG, SD and SV [24] methods. Moreover, several new schemes with favorable properties are discovered in the FR family. Later, Huynh [25] extended the FR framework to handle the diffusion equation, which laid the foundation for the extension to the Navier–Stokes equations. In 2009, Wang and Gao [26,27] extended the flux reconstruction approach to simplex elements under the lifting collocation penalty (LCP) framework. Since the FR and LCP approaches result in the same final formulation, Huynh and Wang renamed FR and LCP schemes to 'correction procedure via reconstruction' or CPR schemes. Further work on the FR/CPR method can be found in [28,29,3,30–33]. A recently developed sparse line-based DG [34] on quadrilateral elements is closely related to these formulations.

In the present study, we evaluate the efficiency, accuracy and robustness of the QDG, NDG, SD and the CPR formulations on quadrilateral elements. Comparison of these formulations on triangular and other types of elements remains to be carried out. Here, accuracy and efficiency are tightly coupled. We measure the relative efficiency based on a given error threshold, following the practice used in the 1st International Workshop on High-Order CFD Methods [35].

The paper is organized as follows. For completeness, we briefly review DG, SD and CPR methods on linear elements in Section 2. Analyses of several schemes on curved elements and the modification of these schemes to enhance accuracy and efficiency are presented in Section 3. In Section 4, the implementation procedures for DG, SD and CPR methods are given and some important assumptions for different implementations are clarified. The comparisons of the algorithm accuracy and computational efficiency for the linear and Euler equations on both linear and high-order elements are presented in Section 5. Finally, conclusions are summarized in Section 6.

2. Reviews of discontinuous high-order methods on linear elements

Consider the following conservation law,

$$\frac{\partial Q}{\partial t} + \nabla \cdot \boldsymbol{F}(Q) = 0 \tag{1}$$

defined on $\Omega \times [0, T]$ with spatial domain Ω bounded by $\partial\Omega$, where Q is the vector of conservative variables, and $\boldsymbol{F} = (f, g)$ is the flux vector, which can be a linear or nonlinear function of Q.

To achieve an efficient implementation, the conservation law is usually transformed from the physical domain (x, y) into the computational domain (ξ, η). Define the area vectors

$$\boldsymbol{a}^1 = (y_\eta, -x_\eta), \qquad \boldsymbol{a}^2 = (-y_\xi, x_\xi), \tag{2}$$

and the volume

$$\tau = x_\xi y_\eta - x_\eta y_\xi, \tag{3}$$

where x_ξ, x_η, y_ξ and y_η are metrics of the coordinate transformation. Then the transformed equation of Eq. (1) takes the following form

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{f}}{\partial \xi} + \frac{\partial \tilde{g}}{\partial \eta} = \frac{\partial \tilde{Q}}{\partial t} + \sum_{l=1}^{2} \frac{\partial \tilde{F}^l}{\partial \xi^l} = 0, \tag{4}$$

where

$$\tilde{Q} = \tau Q, \qquad \left(\tilde{F}^1, \tilde{F}^2\right) = (\tilde{f}, \tilde{g}) = \left(\boldsymbol{a}^1 \cdot \boldsymbol{F}, \boldsymbol{a}^2 \cdot \boldsymbol{F}\right), \qquad \left(\xi^1, \xi^2\right) = (\xi, \eta). \tag{5}$$

It is assumed that the physical domain Ω is partitioned into N non-overlapping elements Ω^i. Note that Eq. (1) holds on each Ω^i. When Ω^i is transformed into the corresponding standard element Ω_s, Eq. (4) holds on Ω_s. Also note that