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Duality based boundary conditions and dual consistent finite difference discretizations of the Navier–Stokes and Euler equations

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ABSTRACT

In this paper we derive new far-field boundary conditions for the time-dependent Navier–Stokes and Euler equations in two space dimensions. The new boundary conditions are derived by simultaneously considering well-posedness of both the primal and dual problems. We moreover require that the boundary conditions for the primal and dual Navier–Stokes equations converge to well-posed boundary conditions for the primal and dual Euler equations.

We perform computations with a high-order finite difference scheme on summation-by-parts form with the new boundary conditions imposed weakly by the simultaneous approximation term. We prove that the scheme is both energy stable and dual consistent and show numerically that both linear and non-linear integral functionals become superconvergent.

1. Introduction

The focus on this paper is the derivation of new boundary conditions for the Navier–Stokes and Euler equations. The technique that will be used is, however, general and can be applied to any initial boundary value problem (IBVP). In particular, we focus on deriving far-field boundary conditions for the Navier–Stokes equations which converge to well-posed boundary conditions the Euler equations in the limit of vanishing viscosity for the difficult subsonic flow case.

There are many numerical methods that can be used to obtain approximate solutions to IBVPs. In this paper, the focus will be on the finite difference method on summation-by-parts (SBP) form with weak implementation of the boundary conditions. The finite difference method is appealing due to its high-order accuracy and ease of implementation.

The finite difference method on SBP form were originally developed by Kreiss and Scherer [19,20] as a means to mimic integration by parts, and to construct high-order accurate energy stable finite difference schemes for linearly well-posed hyperbolic problems. The implementation of the boundary conditions were made feasible by Carpenter et al. in [5] by adding the boundary conditions as penalty terms, the so-called simultaneous approximation terms (SATs). The combination of SBP and SAT allows for energy stable finite difference discretizations of any linearly well-posed IBVP which is independent of the order of accuracy. Details on the construction and properties of the SBP operators can be found in [39] for the first derivative and in [24,22] for the second derivative.

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The SBP-SAT technique has been extended to include curvilinear coordinate transforms [32,42], multi-block couplings [6,31,7,34,23,28], artificial dissipation operators [26,8], and has been applied to numerous applications where it has proven to be robust. See for example [44,25,14,16].

The most recent development of the SBP-SAT technique were made by Hicken and Zingg [13,12]. They analyzed the properties of the discrete norm and showed that superconvergent linear volume integral functionals of linear IBVPs could be computed from so-called dual consistent SBP-SAT discretizations. In general, the solution to an IBVP with a diagonal norm is accurate of order $p+1$ since the interior accuracy is $2p$ with pth-order boundary accuracy [45]. It was shown in [12], and later extended in [3], that linear integral functionals from a diagonal norm dual consistent SBP-SAT discretization retains the full accuracy of $2p$. Dual consistency is a matter of choosing the coefficients in the SATs and does not increase the computational complexity. Superconvergence of linear integral functionals hence comes for free. This theory is still new and work remains to be done for non-linear IBVPs, non-linear functionals, boundary integrals such as lift and drag, and higher-dimensional problems.

Free superconvergence is an attractive property of a dual consistent SBP-SAT discretization. The duality concept can, however, also be used to construct new boundary conditions for the continuous problem. By having advanced boundary conditions for the continuous problem, the numerical scheme can be greatly enhanced [29,30,4].

The superconvergence property have only been proven for linear problems with linear integral functionals. In this paper we will use the linearized Navier–Stokes and Euler equations to derive schemes which will be applied to the non-linear problems and non-linear integral functionals. The linear theory will hence be applied to non-linear problems for which the theory is still incomplete. It was shown in [3] that already for a coarse mesh, the accuracy of linear integral functionals of dual consistent schemes were superior to the dual inconsistent case, even though the solutions were of the same accuracy.

The aim of this paper is threefold. The first goal is to use duality to construct advanced far-field boundary conditions for the Navier–Stokes and Euler equations in two space dimensions. The second is to use the new boundary conditions in an SBP-SAT discretization and show that dual consistency and energy stability becomes equivalent The third is to show computationally that integral functionals are superconvergent even when applied to non-linear problems.

2. The Navier–Stokes equations

The two-dimensional time-dependent compressible Navier–Stokes equations in non-dimensional form can be written as

$$q_t + F^I_x + G^I_y = \varepsilon\left(F^V_x + G^V_y\right), \tag{1}$$

where $\varepsilon = Ma/Re$ is the ratio between the Mach and Reynolds number, $q = [\rho, \rho u, \rho v, e]^T$ are the conservative variables and ρ, u, v, e are the density, velocities, and energy, respectively. The energy is defined as

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\left(u^2 + v^2\right), \tag{2}$$

where γ is the ratio of specific heats and p is the pressure. We assume an ideal fluid and hence the equation of state is

$$\gamma p = \rho T, \tag{3}$$

where T is the temperature. The above variables have been non-dimensionalized using

$$\begin{aligned} \rho &= \frac{\rho^*}{\rho^*_\infty}, \qquad u = \frac{u^*}{c^*_\infty}, \\ v &= \frac{v^*}{c^*_\infty}, \qquad e = \frac{e^*}{\rho^*_\infty (c^*_\infty)^2}, \\ p &= \frac{p^*}{\rho^*_\infty (c^*_\infty)^2}, \qquad T = \frac{T^*}{T^*_\infty}, \end{aligned} \tag{4}$$

where the $*$-superscript denotes a dimensional variable and the ∞-subscript the free stream reference value. The inviscid fluxes are given by

$$F^I = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho u v \\ (p+e)u \end{bmatrix}, \qquad G^I = \begin{bmatrix} \rho v \\ \rho u v \\ p + \rho v^2 \\ (p+e)v \end{bmatrix}, \tag{5}$$

and the viscous fluxes are given by

$$F^V = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + \frac{\kappa}{Pr(\gamma-1)} T_x \end{bmatrix}, \qquad G^V = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ u\tau_{yx} + v\tau_{yy} + \frac{\kappa}{Pr(\gamma-1)} T_y \end{bmatrix}, \tag{6}$$