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Well-balanced schemes for the Euler equations with gravitation

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ABSTRACT

Well-balanced high-order finite volume schemes are designed to approximate the Euler equations with gravitation. The schemes preserve discrete equilibria, corresponding to a large class of physically stable hydrostatic steady states. Based on a novel local hydrostatic reconstruction, the derived schemes are well-balanced for any consistent numerical flux for the Euler equations. The form of the hydrostatic reconstruction is both very simple and computationally efficient as it requires no analytical or numerical integration. Moreover, as required by many interesting astrophysical scenarios, the schemes are applicable beyond the ideal gas law. Both first- and second-order accurate versions of the schemes and their extension to multi-dimensional equilibria are presented. Several numerical experiments demonstrating the superior performance of the well-balanced schemes, as compared to standard finite volume schemes, are also presented.

1. Introduction

1.1. Systems of balance laws

Many interesting physical phenomena are modeled by the Euler equations with gravitational source terms. Examples include the study of atmospheric phenomena that are essential in numerical weather prediction and in climate modeling as well as in a wide variety of contexts in astrophysics such as modeling solar climate or simulating supernova explosions. The Euler equations with gravitational source terms express the conservation of mass, momentum and energy and are given by,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1.1}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v} \rho \boldsymbol{v}) + \nabla p = -\rho \nabla \phi, \tag{1.2}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[(E + p)\boldsymbol{v}\right] = -\rho \boldsymbol{v} \cdot \nabla \phi. \tag{1.3}$$

Here, ρ is the mass density, $\boldsymbol{v}$ the velocity and

$$E = \rho e + \frac{\rho}{2} v^2,$$

the total energy being the sum of internal and kinetic energy. The pressure p is related to the density and specific internal energy through an equation of state $p = p(\rho, e)$.

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The right hand side of the momentum and energy conservation equations models the effect of gravitational forces onto the conserved variables in terms of the gravitational potential ϕ. This potential can be a given function such as the linear gravitational potential, $\phi(x, y, z) = gz$ (with z being the vertical spatial coordinate) that arises in atmospheric modeling or it can also be determined by the Poisson equation

$$\nabla^2\phi = 4\pi G\rho, \tag{1.4}$$

in the case of self-gravity, which is very relevant in an astrophysical context.

The Euler equations with gravitation (1.1)–(1.3) are a prototypical example for a system of balance laws,

$$\mathbf{U}_t + \nabla \cdot \big(\mathbf{F}(\mathbf{U})\big) = \mathbf{S}(\mathbf{U}), \tag{1.5}$$

with $\mathbf{U}, \mathbf{F}$ and $\mathbf{S}$ being the vector of unknowns, the flux and the source, respectively. The special case of $\mathbf{S} \equiv 0$ is termed a conservation law. It is well known [1] that solutions of systems of conservation (balance) laws contain discontinuities in the form of shock waves and contact discontinuities, even when the initial data are smooth. Hence, the solutions of the balance law (1.5) are interpreted in the sense of distributions. Furthermore, these weak solutions may not be unique. Additional admissibility criteria or *entropy conditions* need to be imposed in order to select the physically relevant solution.

Numerical methods for system of conservation (balance) laws are in a mature stage of development. Among the most popular discretization frameworks are the so-called finite volume methods [2], where the cell averages of the unknown are evolved in terms of numerical fluxes. The numerical fluxes are determined by approximate or exact solutions of Riemann problems at each cell interface, in the normal direction. Higher order spatial accuracy is provided by suitable non-oscillatory reconstruction procedures such as TVD, ENO or WENO reconstructions. An alternative for high-order spatial accuracy relies on the Discontinuous Galerkin (DG) finite element method. Higher order time integration is performed by using strong stability preserving (SSP) Runge–Kutta methods. This framework has resulted in efficient resolutions of very complex physical phenomena modeled by system of conservation (balance) laws.

1.2. Role of steady states

An interesting issue that arises when the analysis and numerical approximation of systems of balance laws (1.5) is performed, is the presence of non-trivial steady states. Such steady states (stationary solutions) are defined by the *flux-source balance*:

$$\nabla \cdot \big(\mathbf{F}(\mathbf{U})\big) = \mathbf{S}(\mathbf{U}). \tag{1.6}$$

A particular example is provided by the hydrostatic steady state for the Euler equations with gravitation (1.1)–(1.3). In this stationary solution, the velocity is zero, i.e. $\boldsymbol{v} \equiv 0$ and the pressure exactly balances the gravitational force:

$$\nabla p = -\rho\nabla\phi. \tag{1.7}$$

The above steady state models the so-called *mechanical equilibrium* and is incomplete to some extent as the density and pressure stratifications are not uniquely specified. Another thermodynamic variable is needed (e.g. entropy or temperature) to uniquely determine the equilibrium. However, arbitrary entropy or temperature profiles may not result in physically stable equilibria. For stability, the gradient of the entropy or the temperature profile must fulfill certain criteria (see e.g. [3]). Two important classes of stable hydrostatic equilibria are characterized by constant entropy, i.e. isentropic, and constant temperature, i.e. isothermal, respectively.

As a concrete astrophysically relevant example of a stable stationary state, we assume constant entropy and use the following thermodynamic relation

$$\mathrm{d}h = T\,\mathrm{d}s + \frac{\mathrm{d}p}{\rho}, \tag{1.8}$$

where h is the specific enthalpy

$$h = e + \frac{p}{\rho}, \tag{1.9}$$

T the temperature and s the specific entropy. Then we can write (1.7) for the isentropic case ($\mathrm{d}s = 0$) as

$$\frac{1}{\rho}\nabla p = \nabla h = -\nabla\phi. \tag{1.10}$$

The last equation can then be trivially integrated to obtain,

$$h + \phi = const. \tag{1.11}$$

The importance of steady states such as the above equilibrium lies in the fact that in many situations of interest, the dynamics is realized as a perturbation of the steady states. As examples, consider the simulation of small perturbations on a