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# The Double Absorbing Boundary method

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ABSTRACT

A new approach is devised for solving wave problems in unbounded domains. It has common features to each of two types of existing techniques: local high-order Absorbing Boundary Conditions (ABC) and Perfectly Matched Layers (PML). However, it is different from both and enjoys relative advantages with respect to both. The new method, called the Double Absorbing Boundary (DAB) method, is based on truncating the unbounded domain to produce a finite computational domain $\Omega$, and on applying a local high-order ABC on *two* parallel artificial boundaries, which are a small distance apart, and thus form a thin non-reflecting *layer*. Auxiliary variables are defined on the two boundaries and inside the layer bounded by them, and participate in the numerical scheme. The DAB method is first introduced in general terms, using the 2D scalar time-dependent wave equation as a model. Then it is applied to the 1D Klein–Gordon equation, using finite difference discretization in space and time, and to the 2D wave equation in a wave guide, using finite element discretization in space and dissipative time stepping. The computational aspects of the method are discussed, and numerical experiments demonstrate its performance.



## 1. Introduction

Wave problems in unbounded or very large domains appear often in various important applications such as underwater acoustics, electromagnetic scattering and solid earth geophysics, to name just a few. One large class of computational methods to handle such problems is based on truncating the unbounded domain by introducing an artificial boundary, thus defining a finite computational domain of interest. Some numerical technique is applied on or near the artificial boundary, to minimize spurious reflection of outgoing waves.

Since the mid 1990s, two powerful techniques have emerged in this context: the use of a high-order Absorbing Boundary Condition (ABC) and the use of a Perfectly Matched Layer (PML). Research on these methods is still very active. In fact, it is remarkable that finding an excellent computational tool to deal with the unboundedness of wave problems, which has started in the early 1970s, is still not a "closed problem". A search in the article archive ISI shows that during the last 5 years, more than 150 papers were published with the words ABC or PML in the title, and more than 1000 published papers indicated ABC or PML as keywords. What stands behind this is the fact that the computational tool sought must be accurate, stable, efficient and easy to implement, and should apply in various configurations and for various governing equations. All the existing computational techniques for unbounded wave problems can still use improvements in some of these aspects. See the review papers [1–4].

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The first PML was devised by Bérenger [5] in 1994 for electromagnetic waves, and since then further developed, analyzed and used in various applications by many authors. See references in the review papers [3,6]. The PML is a layer adjacent to the artificial boundary, in which the governing equations are artificially modified. It possesses two properties at the continuous level: (a) there is a perfect match between the layer and the interior domain, namely any outgoing plane wave produces zero reflection; and (b) the solution decays exponentially when it travels inside the layer. These two properties theoretically guarantee excellent performance of the PML. What may sometimes hamper this theoretical performance is the sensitivity of the PML to the discretization and the need to introduce ad-hoc damping and stretching profiles.

The first high-order ABC was devised by Collino [7] in 1993, and a few other formulations followed by other authors. See references in the review paper [8]. High-order ABCs are local in space and time, like the classical ABCs of Engquist and Majda [9] and Bayliss and Turkel [10], but unlike those, they do not involve high derivatives. Therefore they can be implemented in practice up to any desired order, as opposed to the classical ones that have been implemented up to second order only. In the high-order ABC scheme, the order of the ABC is simply an input parameter. The high derivatives that initially appear when designing a high-order ABC are eliminated by introducing auxiliary variables $\phi_j$ on the boundary.

Recently the two approaches — high-order ABC and PML — have been compared, theoretically and numerically [4,11], in the frequency domain. They were found to be equally effective, with some relative advantages for both. In fact, although usually derived by very different approaches, recent work has shown that, at the discrete level, the two methods are quite closely related. In particular, it is shown in [12] how to design a non-standard PML with a purely imaginary mesh continuation to exactly annihilate propagating waves at any incidence angle. This non-standard PML is formally equivalent to the high-order ABC proposed by Hagstrom and Warburton [13].

As mentioned above, each of the two classes of techniques has relative advantages. One major disadvantage of high-order ABCs is that they require special treatment at *corners* formed by the intersection of two flat segments of the artificial boundary, and in some cases also at corners between an artificial and a physical boundary. Such special treatment is sometimes cumbersome or even difficult to devise. In contrast, handling corners with PMLs is usually straight forward. Another disadvantage of high-order ABCs is that they are constrained not to include any *normal derivative* of an auxiliary variable $\phi_j$ on the boundary, since the $\phi_j$ are discretized in practice only on the boundary. Thus, the ABC is allowed to involve only tangential and temporal derivatives of the $\phi_j$. Eliminating the normal derivatives from the ABC operators is sometimes difficult and may require a lot of algebra; a case in point is elastodynamics [14]. PML is also usually easier to incorporate in an existing numerical code.

On the other hand, an important disadvantage of PML is that it is not associated with a clear notion of convergence, except under the expensive scenario of widening a layer where all physical and auxiliary fields are well-resolved. By contrast, in the case of high-order ABCs, with a fixed location of the boundary, one can approach the exact solution arbitrarily closely (up to the discretization error) by increasing the order $P$ of the ABC (with cost that increases only linearly with $P$). More efficient underresolved PMLs seem to be more sensitive to discretization and to the computational parameters than ABCs. A good design of an ABC at the continuous level usually guarantees good performance at the discrete level. This does not seem to be the general case for PML, where the matching between the solutions in the interior and in the layer at the discrete level is sometimes far from perfect. In addition, the theoretical analysis of a PML is usually more difficult than that for a high-order ABC for the same application. Additional discussion on the comparison of the two types of methods can be found in [3,4,11].

In this paper we present a new method, which shares some features of both the PML and the high-order ABC, but enjoys some of the advantages that each of them lacks. In the new method, called the Double Absorbing Boundary method, or simply DAB, a high-order ABC is applied on *two* parallel artificial boundaries, which are a small distance apart. Auxiliary variables are defined on the two boundaries and in the thin layer bounded by them. Like the PML, the DAB does not require special treatment of corners. The algebra involved is quite simple, since no elimination of normal derivatives is needed. Like in the method of high-order ABCs on a single boundary, DAB is clearly associated with the notion of convergence; one can approach the exact solution arbitrarily closely (up to the discretization error) by increasing the order $P$, with only linearly-increasing cost. The numerical properties of DAB, like accuracy, stability and sensitivity to discretization, are similar to those of a high-order ABC on a single boundary.

Although DAB is a general approach, and in principle can be used with any high-order ABC applied on the double boundary, here we consider ABCs of the form proposed by Hagstrom and Warburton (H-W) in [13,15]. The ABC formulation in [15], called the Complete Radiation Boundary Condition (CRBC), generalizes that in [13], and leads to an almost uniform-in-time error estimate for both propagating and decaying waves. The H-W ABCs have been incorporated in both finite element and finite difference schemes, and have been shown to be extremely effective in a variety of situations, including those associated with dispersive, stratified, anisotropic and convective media [16–18], and where exterior sources are present (nesting) [19]. Recently a H-W type ABC was applied to problems in elastodynamics [20,21]. This is the first known high-order ABC for elastodynamics that is long-time stable.

Here is an outline of the remaining sections. In Section 2 we introduce DAB in general terms, using the 2D scalar time-dependent dispersive wave equation (Klein–Gordon equation) as a model. In doing so we comment on the comparison of DAB to existing methods. We analyze the accuracy and stability of DAB in Section 3. In Section 4 we apply DAB to the 1D dispersive wave equation, which is perhaps the simplest (but non-trivial) model that still brings to light the main properties of the method, and allows us to discuss computational aspects in detail. We use Finite Difference (FD) discretization in space and time. We present some numerical experiments for this problem in Section 5, which demonstrate the performance of the