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# Interpolation methods and the accuracy of lattice-Boltzmann mesh refinement

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ABSTRACT

A lattice-Boltzmann model to solve the equivalent of the Navier–Stokes equations on adaptively refined grids is presented. A method for transferring information across interfaces between different grid resolutions was developed following established techniques for finite-volume representations. This new approach relies on a space–time interpolation and solving constrained least-squares problems to ensure conservation. The effectiveness of this method at maintaining the second order accuracy of lattice-Boltzmann is demonstrated through a series of benchmark simulations and detailed mesh refinement studies. These results exhibit smaller solution errors and improved convergence when compared with similar approaches relying only on spatial interpolation. Examples highlighting the mesh adaptivity of this method are also provided.



## 1. Introduction

Over the past twenty years, the lattice-Boltzmann method (LBM) has emerged as a viable alternative to traditional Navier–Stokes solvers [26]. Though it can be derived as a numerical approximation to the classic Boltzmann equation, the LBM has its origins in lattice-gas automata and recovers hydrodynamic behavior in the limit of small Mach and Knudsen numbers. Among its advantages are an efficient algorithm which is straightforward to parallelize, the ease in which complex boundaries can be incorporated, and the possibility to guarantee numerical stability by the implementation of an H-function. The LBM has been successfully applied to variety of flows, including turbulence [32], porous media [11], and hemodynamics [34].

As with any grid-based method, the cost of a lattice-Boltzmann simulation is dictated by the length scale of the smallest features one wishes to resolve. The standard LBM is formulated on a uniform Cartesian grid in either 2-D or 3-D which limits the accessible range of scales and the complexity of flows it can address. Several extensions to the standard LBM have been developed to enable computation on more complex meshes, including curvilinear [22], unstructured [23], and locally refined grids [16]. Of these approaches, mesh refinement, with domains that consist of a hierarchy of uniform grids, is the most attractive since it preserves the simplicity of the original algorithm by avoiding some of the more elaborate transformations required for unstructured meshes [23]. For the LBM, there are two mesh refinement approaches, node-based, in which the grid points are located at cell corners, and therefore the coarse and a fraction of the fine grid points are co-located, and volume-based, where grid points are cell-centered and are not co-located at any level of refinement. At coarse-fine grid

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interfaces, appropriate interpolation and reconstruction algorithms must be constructed to properly manage the flow of information from one grid to another while satisfying conservation laws.

In the node-based approach, first developed by Filippova and Hanel [16], continuity between the fine and coarse grids is achieved by rescaling the non-equilibrium component of the distribution function. Improvements to their method removed a potential singularity [15], whereas Kandhai et al. [18] proposed rescaling the lattice velocities to couple the meshes for a finite-difference implementation of the LBM. Others adopted the distribution rescaling to multi-block methods, which lack any underlying coarse cells [24,31]. Rescaling the distribution function does locally preserve mass and momentum between grids at co-located nodes but does not ensure global conservation, in which the total mass and momentum leaving the coarse grid is equal to the mass and momentum entering the fine grid, and vice-versa [20].

Alternatively, the volume-based methods [6,25] can be formulated to conserve mass and momentum across grid interfaces. Another advantage of the volume-based methods is the ability to incorporate any type of collision operator, whereas the node-based methods are limited to the single relaxation time BGK collision operator [25]. Both piecewise constant [25] and piecewise linear [6] spatial interpolation schemes have been demonstrated to transfer distribution function information from coarse to fine grids, while fine values streaming to the coarse grid are averaged onto the coarse cells.

This paper introduces an alternative cell-centered mesh refinement approach based on the methodology of Colella and Berger [3] for finite-volume computational fluid dynamics (CFD) methods. We propose a space–time interpolation to populate a single layer of ghost cells on each fine grid. We demonstrate that it is fully conservative in mass and more accurate than previous interpolations. We also highlight some of the special considerations that are unique to lattice-Boltzmann grid refinement. Since the original LBM is not a formulation based on control-volumes, there is no flux defined at the faces of the cells surrounding the lattice sites. The consequence of this is that interpolations to fill fine grid ghost cells from the coarse grid need to be fully conservative as conservation cannot be enforced by simply guaranteeing a consistent flux. Another important difference is that in a finite-volume scheme, the fluxes only need to be formulated on the faces of the cells. In contrast, lattice-Boltzmann methods require interpolation and mechanisms for ensuring conservation at each lattice velocity direction near the interface between coarse and fine grids.

As with all AMR schemes, proper care in the construction of the interpolation between grids will ensure the error introduced by the refinement does not excessively affect the solution error. While some lattice-Boltzmann grid refinement efforts discuss error in a limited context, to our knowledge, there has not been a systematic convergence study of refinement errors along with an examination of how the order of the LBM is affected. We describe and present a thorough error analysis using a series of benchmark calculations to validate our approach and to provide a comparison to other methods.

We describe the single grid lattice-Boltzmann algorithm in Section 2, followed by a detailed discussion of our multi-level AMR algorithms in Section 3. Several benchmark simulations, including transient Poiseuille flow, the Taylor–Green vortex array, an acoustic pulse, and laminar vortex shedding behind a cylinder are presented in Section 4. Section 5 contains our summary and conclusions.

## 2. Lattice-Boltzmann method

### 2.1. Single-level algorithm

The problem domain is discretized using a grid, $\Gamma \subset \mathbb{Z}^{\mathbf{D}}$, that is a bounded subset of the integer lattice defined by the points $(j_0, \ldots, j_{\mathbf{D}-1}) = \boldsymbol{j} \in \mathbb{Z}^{\mathbf{D}}$ marking the lattice sites. On a Cartesian grid, the cells around each lattice site take the form

$$V_{\boldsymbol{j}} = \left[\boldsymbol{x}_0 + \left(\boldsymbol{j} - \frac{1}{2}\boldsymbol{u}\right)\Delta x, \boldsymbol{x}_0 + \left(\boldsymbol{j} + \frac{1}{2}\boldsymbol{u}\right)\Delta x\right], \tag{1}$$

where $\boldsymbol{x}_0 \in \mathbb{R}^{\mathbf{D}}$ is some fixed origin of coordinates, $\Delta x$ is the mesh spacing, and $\boldsymbol{u} \in \mathbb{Z}^{\mathbf{D}}$ is the vector whose components are all equal to one.

The lattice-Boltzmann equation describes the evolution of a fictitious single-particle distribution function, $f_i(\boldsymbol{j}, t)$, which is the mass density of particles at time $t$ moving with lattice velocity $\boldsymbol{e}_i$ at site $\boldsymbol{j}$ in $\Gamma$,

$$f_i(\boldsymbol{j} + \boldsymbol{e}_i\Delta t, t + \Delta t) = f_i(\boldsymbol{j}, t) + \mathcal{L}_{ij}\left(f_j(\boldsymbol{j}, t) - f_j^{eq}(\boldsymbol{j}, t)\right) \equiv \hat{f}_i(\boldsymbol{j}, t). \tag{2}$$

The right side of this equation describes the collision process in which the distribution function relaxes to a local equilibrium $f_i^{eq}$, where $\mathcal{L}_{ij}$ is the linear collision operator and $\hat{f}_i(\boldsymbol{j}, t)$ denotes the post-collision state. The most popular version of LB uses the Bhatnagar–Gross–Krook (BGK) collision operator [5], $\mathrm{diag}(\mathcal{L}) = -1/\tau_\nu$, with a single relaxation time, $\tau_\nu$. This parameter is related to the kinematic viscosity by $\nu = c_s^2(\tau_\nu - \frac{1}{2})\Delta t$, where $c_s = \Delta x/\sqrt{3}\,\Delta t$ is the speed of sound, a fixed quantity describing the rate at which information propagates across the lattice. Thus, Eq. (2) can be decomposed into a collision step followed by a streaming step, in which the distribution functions are advanced to neighboring lattice sites in their respective directions determined by $\boldsymbol{e}_i\Delta t$. The macroscopic hydrodynamic quantities, density ($\rho$), momentum ($\rho\mathbf{u}$), and momentum flux ($\Pi_{mn}$), are computed from moments of the distribution functions,

$$\rho = \sum_i f_i, \qquad \rho\boldsymbol{u} = \sum_i f_i\boldsymbol{e}_i, \qquad \Pi_{mn} = \sum_i f_i\boldsymbol{e}_i^m\boldsymbol{e}_i^n. \tag{3}$$