Journal of Computational Physics 259 (2014) 598–616

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A numerical investigation of velocity–pressure reduced order models for incompressible flows

Alfonso Caiazzo [a], Traian Iliescu [b,1], Volker John [a,c,*], Swetlana Schyschlowa [a]

[a] Weierstrass Institute for Applied Analysis and Stochastics, Leibniz Institute in Forschungsverbund Berlin e. V. (WIAS), Mohrenstr. 39, 10117 Berlin, Germany
[b] Department of Mathematics, Virginia Tech, 456 McBryde Hall, Blacksburg, VA 24061-0123, USA
[c] Free University of Berlin, Department of Mathematics and Computer Science, Arnimallee 6, 14195 Berlin, Germany

ARTICLE INFO

Article history:
Received 31 January 2013
Received in revised form 20 November 2013
Accepted 2 December 2013
Available online 11 December 2013

Keywords:
Navier–Stokes equations
Proper orthogonal decomposition
Velocity–pressure reduced order models
Snapshot accuracy

ABSTRACT

This report has two main goals. First, it numerically investigates three velocity–pressure reduced order models (ROMs) for incompressible flows. The proper orthogonal decomposition (POD) is used to generate the modes. One method computes the ROM pressure solely based on the velocity POD modes, whereas the other two ROMs use pressure modes as well. To the best of the authors' knowledge, one of the latter methods is novel. The second goal is to numerically investigate the impact of the snapshot accuracy on the results of the ROMs. Numerical studies are performed on a two-dimensional laminar flow past a circular obstacle. Comparing the results of the ROMs and of the simulations for computing the snapshots, it turns out that the latter results are generally well reproduced by the ROMs. This observation is made for snapshots of different accuracy. Both in terms of reproducing the results of the underlying simulations for obtaining the snapshots and of efficiency, the two ROMs that utilize pressure modes are superior to the ROM that uses only velocity modes.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain and let $[0,T]$ be a finite time interval. Incompressible flows are modeled by the incompressible Navier–Stokes equations (in dimensionless form) for the velocity $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}^d$ and the pressure $p : (0,T] \times \Omega \to \mathbb{R}$

$$\begin{aligned} &\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } (0,T] \times \Omega, \\ &\nabla \cdot \mathbf{u} = 0 \quad \text{in } [0,T] \times \Omega, \end{aligned} \tag{1}$$

where $\mathbf{f}$ models body forces acting on the flow and ν is the inverse of the Reynolds number. System (1) has to be equipped with an initial velocity $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$ and with appropriate boundary conditions on the boundary $\partial\Omega$ of Ω. For the concrete flow problem considered in this report, there is no forcing term ($\mathbf{f} = \mathbf{0}$) and the boundary can be decomposed as

* Corresponding author at: Weierstrass Institute for Applied Analysis and Stochastics, Leibniz Institute in Forschungsverbund Berlin e. V. (WIAS), Mohrenstr. 39, 10117 Berlin, Germany.

E-mail addresses: alfonso.caiazzo@wias-berlin.de (A. Caiazzo), iliescu@vt.edu (T. Iliescu), volker.john@wias-berlin.de (V. John), Swetlana.Schyschlowa@wias-berlin.de (S. Schyschlowa).

[1] Partially supported by the National Science Foundation (DMS-0513542 and OCE-0620464) and the Air Force Office for Scientific Research (FA9550-08-1-0136).

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.12.004

$\partial\Omega = \Gamma_{\rm in} \cup \Gamma_0 \cup \Gamma_{\rm out}$, where the boundary parts are mutually disjoint. Problem (1) is completed with the following boundary conditions:

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{at } [0,T] \times \Gamma_{\rm in} \text{ inlet},$$
$$\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \quad \text{at } [0,T] \times \Gamma_0 \text{ solid walls},$$
$$(\nu\nabla\mathbf{u} - pI)\mathbf{n} = \mathbf{0} \quad \text{at } [0,T] \times \Gamma_{\rm out} \text{ outlet},$$

where $\mathbf{n}$ denotes the outer normal unit vector on $\partial\Omega$.

In order to compute a numerical approximation of the solution of (1) with a finite element method, (1) can be transformed into a time-continuous variational formulation, using the spaces

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega) \colon \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{\rm in} \cup \Gamma_0\}, \qquad Q = L^2(\Omega).$$

Furthermore, let $(\cdot,\cdot)$ denote the standard inner product in $L^2(\Omega)$ and let $\mathbf{u_g} \in H^1(\Omega)$ be an extension of $\mathbf{g}$ into Ω for all t. Then, the time-continuous variational formulation reads: find $\mathbf{u} : [0,T] \to H^1(\Omega)$, such that $\mathbf{u} - \mathbf{u_g} \in \mathbf{V}$ for all t, and $p : (0,T] \to Q$ such that

$$\begin{aligned} &(\partial_t\mathbf{u},\mathbf{v}) + (\nu\nabla\mathbf{u},\nabla\mathbf{v}) + \big((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\big) - (\nabla\cdot\mathbf{v}, p) = 0 \quad \forall\mathbf{v} \in \mathbf{V}, \\ &-(\nabla\cdot\mathbf{u}, q) = 0 \quad \forall q \in Q, \end{aligned} \tag{2}$$

and $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$.

In finite element methods, the spaces $(\mathbf{V}, Q)$ in (2) are replaced by finite-dimensional spaces $(\mathbf{V}^h, Q^h)$ consisting of piecewise polynomials with respect to a triangulation $\mathcal{T}^h$ of Ω. Usually, $(\mathbf{V}^h, Q^h)$ are equipped with a local basis, i.e., with a basis where each basis function has a small support such that an easy construction of the spaces $(\mathbf{V}^h, Q^h)$ is possible.

The use of finite element methods for the numerical solution of (2) allows to compute more and more details of the flow field by increasing the dimension of the finite element spaces. However, the number of basis functions can become very large, yielding large linear or nonlinear systems to be solved in the simulations. Consequently, the numerical simulation of the flow can be very time-consuming. In addition, the finite element basis is generally defined independently of the solution, and it only depends on the structure of the computational mesh. In the case that a priori information on the solution is available, one could transfer this knowledge to the finite element space by pre-adapting the triangulation of Ω.

Reduced order models (ROMs) aim at reducing the computational cost of full finite element, finite difference, or finite volume simulations by drastically reducing the dimension of the solution space. The key idea of ROMs consists in utilizing basis functions that already represent the most important features of the solution. In contrast to finite element bases, ROM bases are global bases. In this report, we focus on ROMs in which the basis functions are obtained through a proper orthogonal decomposition (POD) of a set of snapshots, see, e.g. [3,5,10–15,19,20,22,34,40,46]. Here, the snapshots will be obtained from detailed numerical simulations. It is worth noticing that generally the snapshots might even come from experimental data [4,20].

This report has the following two main goals. First, it investigates three different types of ROMs that compute, besides the velocity, also the pressure, called here for shortness vp-ROMs. One of these vp-ROMs is, to the best of the authors' knowledge, new. Second, this report investigates the impact of the accuracy of the simulations for computing the snapshots, shortly denoted by snapshot accuracy, on the vp-ROM results. The motivation and background for these two numerical investigations are presented next.

To motivate the use of vp-ROMs, we note that although most ROMs for incompressible flows do not include a pressure component, there are important settings in which vp-ROMs are appropriate. From the practical point of view, the pressure is needed in many computational fluid dynamics applications, e.g., for the simulation of fluid–structure interaction problems and for the computation of relevant quantities, such as drag and lift coefficients on solid bodies, and for ROM simulations of shear flows [36]. Other reasons for including the pressure are connected to the definition of ROMs. Using only a velocity ROM leads to a comparatively simple model that can be simulated very efficiently. The rationale behind velocity ROMs, as it can be found in the literature, is that all snapshots are divergence-free, hence all basis functions are divergence-free and consequently the ROM velocity is divergence-free, such that the pressure (which acts as a Lagrange multiplier of the divergence-free constraint) is not needed. As it will be clarified in Section 3.1, the same rationale can be applied in the context of finite element methods and discretely divergence-free velocity fields. In this case, only the integrals of the product of the velocity divergence and all test functions from the discrete pressure space vanish. In fact, many numerical methods for computing the snapshots do not provide pointwise divergence-free flow fields. Even for finite element methods, the discretely divergence-free property does not hold for many popular discretizations of the Navier–Stokes equations. Such examples include the case of using the same finite element spaces for velocity and pressure, where a numerical stabilization becomes necessary, or pressure-correction schemes without reconstructing the discretely divergence-free solution. Experimental data will generally not be divergence-free as well. Altogether, the violation of the divergence-free constraint on the snapshots is another reason for incorporating the pressure into ROMs for incompressible flow simulations. Moreover, as already pointed out in [8], the availability of the pressure enables the computation of the residual of the strong form of