Journal of Computational Physics 259 (2014) 617–635

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A level set two-way wave equation approach for Eulerian interface tracking

Henri Samuel [a,b,c,*]

[a] CNRS, IRAP, 14, avenue Édouard Belin, F-31400 Toulouse, France
[b] Université de Toulouse, UPS-OMP, IRAP, Toulouse, France
[c] Bayerisches Geoinstitut, Universität Bayreuth, Germany

ARTICLE INFO

Article history:
Received 11 June 2013
Received in revised form 16 October 2013
Accepted 23 November 2013
Available online 1 December 2013

Keywords:
Advection
Two-way wave equation
Level set
Interface tracking

ABSTRACT

We present a new approach to perform Eulerian level set interface tracking. It consists in advecting the level set function using a second-order two-way wave equation instead of the standard one-way wave advection equation. The resulting numerical schemes are simple to implement, more stable, and significantly less prone to dissipation errors than popular (*e.g.*, WENO) discretizations of the one-way advection equation of higher order, in particular for long advection times. While both the two-way wave advection and the associated numerical schemes were derived previously, these approaches have never been combined with level set advection. Since the level set function to advect is smooth by construction, this ensures the stability of the solution when using the two-way advection equation discretized using centered finite difference schemes. Our numerical tests show that the two-way wave equation approach yields more accurate results than the standard one-way wave equation for the level set advection, at a considerably smaller computational cost.

1. Introduction

High-accuracy modeling of the advection of a scalar field, C, in an arbitrary external velocity field $\mathbf{u}$ is a fundamental requirement and a recurrent problem in computational fluid dynamics (*e.g.*, [19,15,22] and references therein). This process can be described in its simplest form by the following linear hyperbolic equation:

$$\partial_t C + \mathbf{u} \cdot \nabla C = 0, \tag{1}$$

where t is the time. The above equation is also known as the *one-way wave equation*. A special case of advection problems is the tracking of an interface Ω that marks the boundary between two fluid regions of arbitrary shape and size. The location of the interface may be defined according to the value of C (*e.g.*, $\Omega \equiv \mathbf{x}_c$ such that $C(\mathbf{x}_c) = c$). If C is discontinuous, solving blindly for Eq. (1) using Eulerian approaches can often be challenging because the presence of sharp variations in C yields the indetermination of partial derivatives appearing in the advection term across the discontinuities. A way around these difficulties is the level set approach that consists in replacing in Eq. (1) the sharply varying field C by a continuous and smoothly varying field ϕ [16]:

$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi = 0. \tag{2}$$

* Correspondence to: CNRS, IRAP, 14, avenue Édouard Belin, F-31400 Toulouse, France
E-mail address: henri.samuel@irap.omp.eu.

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.11.027

The level set function, ϕ, is usually defined as the signed distance function to the interface that satisfies the Eikonal equation:

$$|\nabla\phi| = 1. \tag{3}$$

In general, the external velocity field $\mathbf{u}$ will not maintain ϕ as a signed distance function, therefore the above Eikonal requirement must be continuously enforced. Failing in doing so may result in the development of non-smooth variations in ϕ, yielding poorly accurate results when solving for Eq. (2). The Eikonal requirement can be enforced by solving for Eq. (3) using a fast marching method [19] or by solving a non-linear evolutionary equation to steady state [21]:

$$\partial_\tau\phi = S(\phi_0)\big(1 - |\nabla\phi|\big), \tag{4}$$

where ϕ_0 is the solution of Eq. (2) prior to reinitialization, τ represents a fictitious time, and $S(\phi_0)$ is a smoothed signed function:

$$S(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 + \varepsilon^2}}, \tag{5}$$

where ε is taken as the grid spacing (assuming a constant grid spacing). Alternatively, one may construct an *extension velocity* field $\mathbf{u}_{\text{ext}}$ that replaces $\mathbf{u}$ in Eq. (1) such that the Eikonal requirement is preserved [1,19]. However, the construction of $\mathbf{u}_{\text{ext}}$ is performed via the fast marching method, which reduces to low order (second at most) compared to the use of higher order (*e.g.*, fifth-order WENO) schemes to discretize the spatial derivatives in Eq. (4). Higher order fast marching methods exist [2] but are more difficult to implement. Since the level set function is smooth, the numerical errors associated with the advection step tend to be smaller than in the case of discontinuous/sharply varying quantities. Nevertheless, even when using high-order schemes such as fifth-order WENO reconstruction, dissipation errors remain present and constitute a main source of inaccuracy. Increasing the order of the advection schemes or using Lagrangian particles to correct for erroneous values of the level set [4,5] considerably reduces the numerical errors. However, such strategies are generally more complicated to implement and to parallelize efficiently. In addition, they yield a substantial extra cost, in particular when the interface spans a significant fraction of the computational domain [18]. It is therefore desirable to devise a level set method that improves the accuracy of the Eulerian advection step and its associated computational cost.

In this paper, we propose to use a second order, *two-way* wave equation instead of Eq. (1) to perform the advection step of the level set method. If the initial conditions are appropriately specified, the solutions of the standard one-way wave equation (Eq. (1)) and the two-way advection equation are identical. However, the discretization of the two-way advection equation yields significantly smaller dissipation and dispersion errors, provided that the function to advect is smooth. The paper is organized as follows: In Section 2 we summarize a few popular approaches to discretize the one-way advection equation (Eq. (1)). In Section 3 we present the two-way advection equation as an alternative to the one-way advection equation. In Section 4 we compare the accuracy of the solution of the one-way advection equation, and the two-way advection equation via theoretical analysis and simple linear advection tests. Section 5 shows the application of the two-way advection equation for level set interface tracking and the comparison with the results obtained using standard level set approach based on the one-way advection equation.

2. The one-way advection equation

In the following we briefly review a few popular approaches to solve the standard advection equation in Eulerian domains. We focus on the one-dimensional version of Eq. (1) along the rectilinear x-direction:

$$\partial_t C + u\partial_x C = 0. \tag{6}$$

Extension to multidimensional space is straightforward and will be presented in Section 2.2.

2.1. Spatial discretization

Assuming a divergence-free velocity field, Eq. (6) can be written in semi-discrete conservative flux difference form:

$$\frac{dC_j}{dt} + \frac{1}{\Delta x}(F_{j+1/2} - F_{j-1/2}) = 0, \tag{7}$$

where Δx is the grid spacing/cell length (assumed to be constant), j is the grid index along the x-direction that refers to the cell-center coordinate, $x_j = \Delta x j$, and $x_{j\pm1/2} = x_j \pm \Delta x/2$. $F_{j\pm1/2}$ is the numerical convective flux through the interface located between cell j and cell $j\pm1$ at time t. For stability, the flux $F_{j\pm1/2}$ is upwind-biased according to the method of characteristics:

$$F_{j\pm1/2} = -\min\big[\text{sign}(u_{j\pm1/2}), 0\big]F^-_{j\pm1/2} + \max\big[\text{sign}(u_{j\pm1/2}), 0\big]F^+_{j\pm1/2}. \tag{8}$$