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A Hermite pseudospectral solver for two-dimensional incompressible flows on infinite domains

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ABSTRACT

The Hermite pseudospectral method is applied to solve the Navier–Stokes equations on a two-dimensional infinite domain. The feature of Hermite function allows us to adopt larger time steps than other spectral methods, but also leads to some extra computation when the stream function is calculated from the vorticity field. The scaling factor is employed to increase the resolution within the region of our main interest, and the aliasing error is fully removed by the 2/3-*rule*. Several traditional numerical experiments are performed with high accuracy, and some related future work on physical applications of this program is also discussed.

1. Introduction

In practice, the boundary effect of flows is inevitable and requires careful treatment. On the other hand, when people concentrate on physical mechanism of fluid, they hope that the boundary effect can be fully removed. For example, the Oseen vortex in two-dimensional (2D) fluid adopts the assumption of the infinite domain (e.g., see [1] and references therein). It has also been shown in many previous investigations on how the boundary can influence the physical mechanism of flows. Even when the region of main interest is fairly far away from the solid boundary, the drop speeds show about 15% difference in a thermocapillary migration study [2]. In the research of the axisymmetrization of an isolated 2D vortex [3], a double-sized computing domain is adopted to alleviate the vortex overrotation caused by the periodical boundary in a Fourier spectral simulation [4]. Hence, it is essential to adopt the infinite domain in such studies. When the infinite computing domain $(-\infty, +\infty)$ is considered, one possible way is still using the finite-domain solver but with a certain mapping between (x_{min}, x_{max}) and $(-\infty, +\infty)$. Another way is something like the sponge-layer suggested in [5], which absorbs the incoming vortex filaments. The other way is more natural: Laguerre or Hermite functions are adopted to construct global approximation to functions defined on unbounded intervals.

The normalized Hermite function of degree n is defined as:

$$\hat{H}_n(x) = \frac{1}{\sqrt{2^n n!}} e^{-\frac{x^2}{2}} H_n(x), \tag{1}$$

where $H_n(x)$ are the usual (unnormalized) Hermite polynomials (for instance, see [6,7]). $\hat{H}_n(x)$ are orthogonal in $L^2(-\infty, \infty)$:

$$\int_{-\infty}^{+\infty} \hat{H}_k(x)\hat{H}_m(x) = \sqrt{\pi}\,\delta_{km}, \quad k, m \geqslant 0.$$

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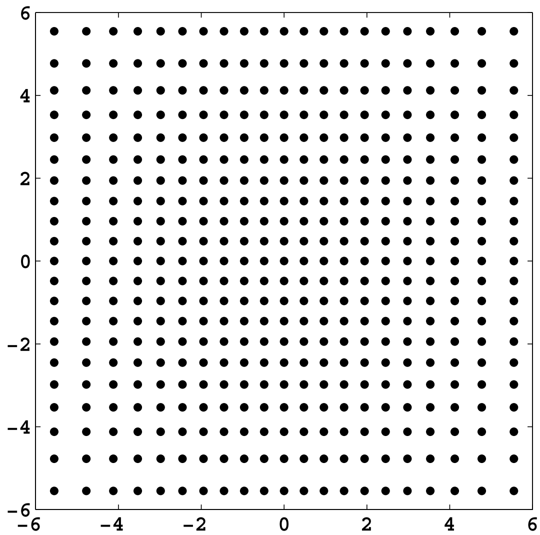

Fig. 1. Distribution of grid points for the resolution of 20×20. Here, $x_0 = 5.55$ is the largest root of $\hat{H}_{21}(x)$. Note that the density of points is slightly higher in the center, which is much clearer in Fig. 2.

In the last few decades, many investigations concerning the theory and application of Hermite functions have been carried out, and an early review can be seen in [8]. Although the computing domain is infinite for Hermite spectral methods, the region of our main interest is still finite. This fact led to the appearance of the most important concept in the practical sense: the scaling factor [9,10]. There have been many Hermite spectral (or, pseudospectral) studies which investigated partial differential equations from different fields [11–20]. Most researches in this field have so far dealt with one-dimensional problems, or multi-dimensional problems with only one Hermite direction. One most recent paper [21] deals with two-dimensional partial differential equations with two Hermite directions, and takes an elliptic equation with a harmonic potential and a class of nonlinear wave equations into consideration. However, so far as we know, there has been no effort in solving the multi-dimensional Navier–Stokes (NS) equations with pure Hermite spectral (or Hermite pseudospectral) methods, and this is the main object of this paper.

The paper is arranged as follows: the numerical details are described in Section 2, and the performance of some validating tests is introduced in Section 3, and the discussions and some related future work are presented in Section 4.

2. Numerical scheme for the Hermite pseudospectral NS solver

The 2D incompressible NS equations on the infinite domain $\vec{x} = (x, y) \in [-\infty, +\infty] \times [-\infty, +\infty]$ in terms of vorticity and stream function are written as

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega, \tag{2}$$

$$\Delta \psi = -\omega, \tag{3}$$

where $\mathbf{u} = (u, \upsilon)$ is the velocity, ν the kinematic viscosity, ψ stream function, and vorticity $\boldsymbol{\omega} = (0, 0, \omega) = \nabla \times \mathbf{u}$. The stream function is related to velocity by $u = \partial\psi/\partial y$ and $\upsilon = -\partial\psi/\partial x$. We use pseudospectral methods to solve Eq. (2) by expanding u, υ, ψ, and ω in a truncated Hermite series.

2.1. Expansions

In pseudospectral methods, the optimal pseudoscpectral points are the roots of $\hat{H}_{N+1}(x)$, which are denoted by $\{x\}_{j=0}^{N}$ with the order $x_0 > x_1 > \cdots > x_N$. Since $x_0 = -x_N \approx \sqrt{2N}$, the values of vorticity in our solver are defined on the grid points in the box of $[-x_0, x_0] \times [-x_0, x_0]$ (e.g., see Fig. 1 or Fig. 6 in [7]).

With the three-term recurrence:

$$\hat{H}_0(x) = e^{-x^2/2}, \qquad \hat{H}_1(x) = \sqrt{2}xe^{-x^2/2},$$

$$\hat{H}_{n+1}(x) = x\sqrt{\frac{2}{n+1}}\hat{H}_n(x) - \sqrt{\frac{n}{n+1}}\hat{H}_{n-1}(x), \quad n \geqslant 1, \tag{4}$$

the values of $\hat{H}_n(x_j)$, $0 \leqslant n, j \leqslant N$ can be calculated. The 2D vorticity field is then transformed to the Hermite spectral space with the following equation: