Journal of Computational Physics 256 (2014) 416–427

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Newton multigrid least-squares FEM for the V-V-P formulation of the Navier–Stokes equations

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ARTICLE INFO

Article history:
Received 12 February 2013
Received in revised form 3 September 2013
Accepted 5 September 2013
Available online 17 September 2013

Keywords:
Least-squares FEM
Newton multigrid
Conjugate gradient
Navier–Stokes

ABSTRACT

We solve the V-V-P, vorticity–velocity–pressure, formulation of the stationary incompressible Navier–Stokes equations based on the least-squares finite element method. For the discrete systems, we use a conjugate gradient (CG) solver accelerated with a geometric multigrid preconditioner for the complete system. In addition, we employ a Krylov space smoother inside of the multigrid which allows a parameter-free smoothing. Combining this linear solver with the Newton linearization, we construct a very robust and efficient solver. We use biquadratic finite elements to enhance the mass conservation of the least-squares method for the inflow–outflow problems and to obtain highly accurate results. We demonstrate the advantages of using the higher order finite elements and the grid independent solver behavior through the solution of three stationary laminar flow problems of benchmarking character. The comparisons show excellent agreement between our results and those of the Galerkin mixed finite element method as well as available reference solutions.

1. Introduction

The least-squares finite element method (LSFEM) is a numerical method for the solution of partial differential equations. The LSFEM is generally motivated by the desire to recover the advantageous features of Rayleigh–Ritz methods, as for instance, the choice of the approximation spaces is free from discrete compatibility conditions and the corresponding discrete system is symmetric and positive definite [1].

In this paper, we solve the incompressible Navier–Stokes (NS) equations with the LSFEM. Direct application of the LSFEM to the second-order NS equations requires the use of quite impractical C^1 finite elements [1]. Therefore, we introduce the vorticity as a new variable to recast the NS equations to a first-order system of equations, i.e. the vorticity–velocity–pressure (V-V-P) system. The classical V-V-P formulation has been investigated by many authors, for instance by Bochev [2], Jiang [3] and Bochev and Gungburger [1] and with further modifications by Heys et al. [4–6]. We study the classical V-V-P system.

As it was mentioned, the resulting LSFEM system is symmetric and positive definite [1]. This permits the use of the conjugate gradient (CG) method and efficient multigrid solvers for the solution of the discrete systems. In order to improve the efficiency of the solution method, the multigrid and the Krylov subspace method, here CG, can be combined with two different strategies. The first strategy is to use the multigrid as a preconditioner for the Krylov method [7]. The advantage of this scheme is that the Krylov method reduces the error in eigenmodes that are not being effectively reduced by multigrid. The second strategy is to employ Krylov methods as multigrid smoother. The Krylov methods appropriately determine the

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0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.09.011

size of the solution updates at each smoothing step [8]. This leads to smoothing sweeps which, in contrast to the standard SOR or Jacobi smoothing, are free from predefined damping parameters.

Heys et al. studied the LSFEM solution of the Stokes equation [7] and the NS equations [9,4,6] with an algebraic multigrid preconditioned CG method. A geometric multigrid preconditioned CG solver was used by Ranjan and Reddy [10] for the Spectral/hp LSFEM solution of the NS equations. They demonstrated superior convergence of the multigrid solver compared to the Jacobi preconditioning. More interestingly, Köster [11] and Wobker [8] used preconditioned BiCGStab as smoother in a geometric multigrid method as well as an outer solver around it to solve the Poisson equation with standard Galerkin finite element method. They reported higher numerical stability and lower total costs of the solution process compared to the stand-alone multigrid or BiCGStab solvers.

We develop a geometric multigrid solver as a preconditioner for the CG (MPCG) iterations to solve the V-V-P system with LSFEM. In addition, we use a CG pre/post-smoother to obtain efficient and parameter-free smoothing sweeps. We demonstrate a robust and grid independent behavior for the solution of different flow problems with both bilinear and biquadratic finite elements.

Despite the advantages of the LSFEM, the lack of local mass conservation of this method is one of its drawbacks. Different strategies have been employed to overcome this deficiency. For very recent techniques and also an overview of the previous efforts we refer to the works of Bochev et al. [12,13]. One remedy for 2D problems, which is also analyzed in this work, is to use higher order finite elements [14]. Weighting the continuity equation more strongly [15] is another well-known method to recover mass conservation. We show, through the Poiseuille flow and the flow around cylinder problems, that quadratic finite elements satisfy the mass conservation to a great extent without the need to further weight the continuity equation.

Moreover, we show that accurate results can be obtained with V-V-P LSFEM provided that higher order finite elements are used. We demonstrate this with a quantitative analysis of the flow around cylinder and the lid-driven cavity problems.

Therefore, the paper is organized as follows: in the next section we introduce the incompressible NS equations, the continuous and the discrete least-square principles with their properties and the designed LSFEM solver. In the next section, we present the general MPCG solver settings and the detailed results of three incompressible fluid flow problems. Finally, we make a conclusion in the last section.

2. LSFEM for the Navier–Stokes equations

2.1. Governing equations

The incompressible NS equations for a stationary flow are given by

$$\begin{cases} \boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p-\nu\Delta\boldsymbol{u}=f & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}=\boldsymbol{g}_D & \text{on } \Gamma_D, \\ \boldsymbol{n}\cdot\boldsymbol{\sigma}=\boldsymbol{g}_N & \text{on } \Gamma_N \end{cases} \tag{1}$$

along with the zero mean pressure constraint

$$\int_\Omega p=0 \tag{2}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain, p is the normalized pressure $p=P/\rho$, $\nu=\mu/\rho$ is the kinematic viscosity, f is the source term, $\boldsymbol{g}_D$ is the value of the Dirichlet boundary conditions on the Dirichlet boundary Γ_D, $\boldsymbol{g}_N$ is the prescribed traction on the Neumann boundary Γ_N, $\boldsymbol{n}$ is the outward unit normal on the boundary, $\boldsymbol{\sigma}$ is the stress tensor and $\Gamma=\Gamma_D\cup\Gamma_N$ and $\Gamma_D\cap\Gamma_N=\emptyset$. The kinematic viscosity and the density of the fluid are assumed to be constant.

The first equation in (1) is the momentum equation where velocities $\boldsymbol{u}=[u,v]^T$ and pressure p are the unknowns and the second equation represents the continuity equation.

2.2. First-order systems

The straightforward application of the LSFEM to the second-order NS equations requires C^1 finite elements [1]. To avoid the practical difficulties in the implementation of such FEMs, we first recast the second-order equation to a system of first-order equations. Another important reason for not using the straightforward LSFEM is that the resulting system matrix will be ill-conditioned [16].

2.2.1. Vorticity–velocity–pressure formulation

A common strategy to reformulate the second-order NS equations to an equivalent first-order system is to introduce the vorticity, ω, as a new variable [1]. In two-dimensional problems the vorticity is a scalar and defined as

$$\omega=\nabla\times\boldsymbol{u}. \tag{3}$$