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A reconstruction algorithm with flexible stencils for anisotropic diffusion equations on 2D skewed meshes

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ABSTRACT

A new reconstruction algorithm is developed to obtain diffusion schemes with cell-centered unknowns only. The main characteristic of the new algorithm is the flexibility of stencils when the auxiliary unknowns are reconstructed with cell-centered unknowns. The stencils are selected depending on the mesh geometry and discontinuities of diffusion coefficients. Moreover, an explicit expression is derived for interpolating the auxiliary unknowns in terms of cell-centered unknowns, and the auxiliary unknowns can be defined at any point on the edge. The algorithm is applied to construct several new diffusion schemes, whose effectiveness is illustrated by numerical experiments. For anisotropic problems with or without discontinuities, nearly second order accuracy is achieved on skewed meshes.

1. Introduction

We present here a new algorithm with flexible stencils for the reconstruction of edge unknowns in two-dimensional space. It can be applied to solve diffusion equations with anisotropic heterogeneous coefficients on skewed meshes. Such problems are commonly encountered in a wide range of scientific fields such as heat transfer, plasma physics, and oil reservoir simulation. In engineering computation, skewed meshes have to be dealt with, for example, in the cases that distorted grids are generated on physical domains with complex geometry.

For an isotropic operator, the linear two-point flux approximation is accurate to second order on rectangular meshes or Voronoi meshes. However, more than two points are needed to maintain the accuracy for anisotropic problems or skewed meshes. The additional points could be adjacent cell-centers, vertex or edge points. In Refs. [1–4], such additional points contribute to calculating the coefficients of discrete flux. A nonlinear two-point flux approximation is then obtained. The final scheme has a compact stencil. It keeps the property of monotonicity, which is desirable in engineering.

In this paper, conventional linear schemes are studied due to its efficiency. In this class of schemes, the vertex or edge unknowns can also be treated as primary unknowns, as has been done in the local support operator method (LSOM) [5], diamond schemes [6,7], the hybrid finite volume scheme [8], etc. These schemes are often more expensive than those with cell-centered unknowns only.

To improve efficiency, the auxiliary unknowns are eliminated by reconstructing them with cell-centered unknowns. One common way is to formulate and solve a local linear system satisfied by auxiliary (edge or vertex) unknowns [9–12]. In the present work, however, we aim to develop a new method. In particular, an explicit expression is given for the reconstruction so as to simplify the code implementation.

In Ref. [13], the least-square method is used for reconstructing vertex unknowns with cell-centered unknowns, where the diffusion tensor, as well as the solution, should be smooth so that the weights can be determined by the least-square method.

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An interpolation-free method is developed for problems with smooth diffusion tensors in Ref. [2]. The method is extended to solving equations with heterogeneous tensor in Ref. [3] and advection-diffusion equations in Ref. [4] on three-dimensional meshes. For interpolating auxiliary unknowns defined at the discontinuities, the authors [3,4] give a new nonlinear two-point approximation method, where the coefficients for the interpolation are always non-negative. Then, a monotone nonlinear scheme is constructed.

A delicate reconstruction method is designed in Ref. [14] which can treat the discontinuity strictly. The authors obtain an explicit expression for the reconstruction by defining the auxiliary unknown at the harmonic averaging point instead of at the midpoint. At the harmonic point, the unknown is reconstructed with a two-point linear interpolation. Such an interpolation is then applied to approximate the gradient, resulting in the SUSHI scheme [15] as well as a small-stencil scheme in three-dimensional space [16]. But on seriously skewed meshes, the harmonic averaging points may be located outside the edges. They are then replaced by the center of an edge, leading to the introduction of additional unknowns.

At the harmonic averaging points, the interpolation coefficients are non-negative, and the sum of the coefficients is one. This is important for preserving the minimum and maximum values of the solution. In Ref. [17], the harmonic averaging points are introduced at the heterogeneous interface. A nonlinear small-stencil scheme is constructed. Note that this scheme satisfies discrete maximum principle.

In Ref. [18], we presented a reconstruction algorithm for isotropic problems. With this algorithm, auxiliary edge unknowns are defined at balance points, the location of which depends on the diffusion coefficients and the skewness of grid cells. Here "balance" means that the position of the point is determined by imposing the continuity condition of the diffusion flux defined there. Note that the position of the balance point coincides with that of the harmonic averaging point in Ref. [14]. But we derived it in a different way and constructed a new scheme. Unlike the SUSHI scheme [15], auxiliary unknowns are defined at the balance points even if they lie outside the edges. Thus our scheme involves cell-centered unknowns only.

In this paper, we investigate the accuracy of schemes using balance points for heterogeneous anisotropic problems. Moreover, a new reconstruction method is proposed in order to provide an alternative to the balance point. Its main benefits are the following:

- An auxiliary unknown can be specified at any point on the edge.
- An explicit expression is obtained for reconstructing auxiliary unknowns with cell-centered unknowns.
- The reconstruction algorithm deals rigorously with discontinuities.
- The reconstruction algorithm involves a flexible stencil whose choice depends on the mesh geometry and discontinuities of diffusion coefficients.
- The reconstruction algorithm reduces to a two-point approximation at a specific point, namely, the balance point.

Several new diffusion schemes are developed. Specifically, we construct two schemes by defining the auxiliary unknowns at the balance points and midpoints, respectively. A hybrid scheme is also presented. In this scheme, the auxiliary unknown is specified at the balance point if it is located inside the edge. Otherwise, the balance point is substituted by the midpoint. An enlarged stencil is adopted for reconstructing the unknown at the midpoint.

The remainder of this paper is organized as follows. First we describe the problems and notation in Section 2. The decomposition of diffusion flux is then given in Section 3. Next, we evaluate auxiliary unknowns using cell-centered unknowns in Section 4 and then approximate the gradient with auxiliary unknowns in Section 5. Several diffusion schemes are formulated in Section 6, followed by numerical examples in Section 7. Finally, we conclude in Section 8.

2. Problem description and notation

Let Ω be an open bounded subset of R^d with $\partial\Omega$ being its boundary. We consider the following diffusion problem

$$-\nabla \cdot \big(\mathcal{D}(\mathbf{x})\nabla u\big) = f \quad \text{in } \Omega, \tag{1}$$

where

- u is a scalar function. In the case of heat conduction, u denotes the temperature. For flows through porous media, u represents the pressure.
- f is the intensity of sources.
- $\mathcal{D}(\mathbf{x}) = (d_{i,j})$ is a given tensor which is symmetric. Moreover, there exists a constant $c > 0$ such that

$$\mathcal{D}(\mathbf{x})\xi \cdot \xi \geqslant c|\xi|^2 \quad \text{for all } \xi \in R^d. \tag{2}$$

We consider boundary conditions of the Dirichlet or Neumann type

$$u = \bar{u}, \quad \text{on } \partial\Omega_D,$$

$$-\mathcal{D}(\mathbf{x})\nabla u \cdot \mathbf{n} = F_2, \quad \text{on } \partial\Omega_N,$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, and $\mathbf{n}$ is the unit outward normal vector.