Journal of Computational Physics 258 (2014) 650–688

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Global series solutions of nonlinear differential equations with shocks using Walsh functions

Peter A. Gnoffo

NASA Langley Research Center, MS 408A, Hampton, VA 23681-2199, USA

ARTICLE INFO

Article history:
Received 9 May 2013
Received in revised form 10 October 2013
Accepted 29 October 2013
Available online 7 November 2013

Keywords:
Closure
Walsh function
Haar wavelet
Shock capturing
Burger's equation
Nozzle
Euler

ABSTRACT

An orthonormal basis set composed of Walsh functions is used for deriving global solutions (valid over the entire domain) to nonlinear differential equations that include discontinuities. Function $g_n(x)$ of the set, a scaled Walsh function in sequency order, is comprised of n piecewise constant values (square waves) across the domain $x_a \leqslant x \leqslant x_b$. Only two square wave lengths are allowed in any function and a new derivation of the basis functions applies a fractal-like algorithm (infinitely self-similar) focused on the distribution of wave lengths. This distribution is determined by a recursive folding algorithm that propagates fundamental symmetries to successive values of n. Functions, including those with discontinuities, may be represented on the domain as a series in $g_n(x)$ with no occurrence of a Gibbs phenomenon (ringing) across the discontinuity. A much more powerful, self-mapping characteristic of the series is closure under multiplication – the product of any two Walsh functions is also a Walsh function. This self-mapping characteristic transforms the solution of nonlinear differential equations to the solution of systems of polynomial equations if the original nonlinearities can be represented as products of the dependent variables and the convergence of the series for $n \to \infty$ can be demonstrated. Fundamental operations (reciprocal, integral, derivative) on Walsh function series representations of functions with discontinuities are defined. Examples are presented for solution of the time dependent Burger's equation and for quasi-one-dimensional nozzle flow including a shock.

Published by Elsevier Inc.

1. Introduction

An orthonormal basis set is introduced for use in generating global solutions to nonlinear differential equations with shocks. By global it is meant that a single series solution is generated that is valid over the entire domain – not a separate solution across discretized elements of the domain. Its derivation was motivated by shortcomings in all current computational fluid dynamic algorithms in dealing with complex, hypersonic flows involving interacting shocks. It will be shown that the fractal-like derivation (infinitely self-similar) focused on the distribution of segment lengths yields a scaled set of Walsh functions [1]. At this point, it is far too early to know if the approach introduced here will alleviate shock related problems without introducing new challenges – perhaps even more difficult than the ones faced today with current state-of-the-art. At a minimum, analysis using these basis functions are expected to provide new insights to guide future algorithm development.

A brief review of the challenges associated with the simulation of shock waves is warranted here because these challenges suggest constraints that are critical to the evolution of the new orthonormal basis function set. Computational fluid

E-mail address: peter.a.gnoffo@nasa.gov.

0021-9991/$ – see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jcp.2013.10.054

dynamic simulation of hypersonic flows generally requires specialized algorithm modifications to accommodate shocks. Two fundamentally different approaches, shock fitting and shock capturing, have evolved to simulate shocks.

The first approach, shock fitting, formally recognizes a shock as a discontinuity in the field [2]. The orientation of the shock and its velocity in the domain are tracked as supplementary dependent variables in the solution. A shock may be tracked as a moving boundary as in the case of blunt body flow or treated as a discontinuity that moves within the interior of a domain. Rankine Hugoniot equations are applied to define dependent variables across the discontinuity. In the most general case, algorithms must be able to detect the formation, interaction, and disappearance of multiple shocks in the domain. Shock fitting approaches produce high accuracy at the expense of high algorithm complexity to accommodate various permutations of interactions. Great advantage may often be realized by fitting only the primary shocks in a domain and using shock capturing to treat any additional discontinuities that may develop in the domain.

The second approach, shock capturing, formulates the governing flow equations in strong conservation form – shocks are assumed to be computable to the extent that the representation of conserved flux is smoothly varying across discontinuities. In practice, flux reconstruction algorithms usually require some capability to detect captured shocks or slip surfaces and employ appropriate limiters to maintain stability and suppress Gibbs phenomenon. Algorithm fixes to suppress the "carbuncle" [3,4] formation in blunt body flows or non-physical instabilities across moving shock fronts tend to be empirically derived – seeking to tune enhanced dissipation in a manner that will prohibit non-physical anomalies without sacrificing solution accuracy [5]. Shock capturing algorithms may employ grid adaptation that in some sense approximates the shock fitting functionality. Shock capturing algorithms will generally require greater grid resources to achieve equivalent accuracy of a shock fitting algorithm assuming smooth regions of the flow are computed similarly.

The initial exploration of segmented basis functions was undertaken to address challenges associated with shock capturing in simulations of hypersonic flows over blunt bodies using tetrahedral grids [6,7]. It is observed that in second-order, finite volume formulations the heating in the stagnation region is poorly computed if the grid is not well aligned with the captured shock. The use of multi-dimensional reconstruction results in significant improvement in stagnation region heating though solution quality is still not as good as obtained with a well-aligned structured (or prismatic) grid. Some of the persisting problems in tetrahedra with multi-dimensional reconstruction are associated with sensitivity to the selection of a primary direction orthogonal to the captured shock. Limiters are engaged as a function of this direction and some disruption of heating is still observed. The detection of a shock-normal direction in a tetrahedral element normally involves calculation of a pressure gradient which itself is a function of element metrics and inherent numerical dissipation. Some of these difficulties have recently been overcome through use of higher-order Discontinuous Galerkin (DG) within the element and application of a PDE-based dissipation that provides much cleaner shock captures [8]. Borrowing from these successes in DG, it was thought that a basis function set composed of simple, discontinuous square waves may enable a more robust algorithm for detecting a shock normal direction even in the context of a finite-volume algorithm. As the orthonormal basis set evolved, it became clear that its self-mapping property under multiplication provided new capabilities for understanding nonlinear problems far beyond its original intent for multi-dimensional reconstruction. The global solution approach developed here solves for variables in wave number space – not physical space. The physical domain is not explicitly discretized although an implicit discretization is engaged as a function of wave number. This approach provides new opportunities and challenges for capturing shocks while retaining their discontinuous structure.

It is noted that the use of Walsh functions for solving nonlinear differential equations has been studied in the past [9, 10]. Both the integrating matrix [11,9] and the self-mapping property under multiplication [10,12] derived herein have been identified previously. The current derivations offer a different perspective, focused on the relation of segment sizes in the basis functions and associated resolution requirements in computational fluid dynamic applications. The ability to accurately locate a discontinuity in a solution or to resolve the structure of a viscous shock wave is a primary focus of the present work; examples are chosen throughout to highlight algorithm requirements (filtering of highest wave number components) to achieve this end.

The paper is organized as follows. The derivation of the orthonormal basis function set is provided in Section 2. This section highlights the self-mapping property under multiplication and discusses similarities and differences with respect to the Haar wavelets. Examples of series representations of both continuous and non-continuous functions appear in Section 3. Sections 4–6 each derive series transformations critical to the solution of nonlinear differential equations. The reciprocal transformation is derived in Section 4; the integral transformation is derived in Section 5; and its inverse transformation, the derivative, is developed in Section 6. Representation of functions of two variables is discussed in Section 7. Finally, examples of the derivation of the global series solution for two nonlinear differential equation sets with shocks are provided to demonstrate that the method indeed works. Solutions for the time-accurate Burger's equation (one dependent variable as a function of space and time) and the quasi-one-dimensional nozzle flow (three dependent variables as a function of space) are presented in this section.

2. Basis function derivation

2.1. Preliminaries

Initial attempts to develop a segmented basis function set started from a constraint that the domain would be divided into segments of equal length and the value of the function on any segment would be allowed to vary from segment