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Weak imposition of the slip boundary condition on curved boundaries for Stokes flow

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ABSTRACT

We study the finite element approximation of two methods to weakly impose a slip boundary condition for incompressible fluid flows: the Lagrange multiplier method and Nitsche's method. For each method, we can distinguish several formulations depending on the values of some real parameters. In the case of a spatial domain with a polygonal or polyhedral boundary, we prove convergence results of their finite element approximations, extending previous results of Verfürth [33] and we show numerical results confirming them. In the case of a spatial domain with a smooth curved boundary, numerical results show that approximations computed on polygonal domains approximating the original domain may not converge to the exact solution, depending on the values of the aforementioned parameters and on the finite element discretization. These negative results seem to highlight Babuska's like paradox, due to the approximation of the boundary by polygonal ones. In particular, they seem to contradict some of Verfürth's theoretical convergence results.

1. Introduction

In Ω, an open bounded and connected subset of $\mathbf{R}^n$, $n=2$ or 3, with Lipschitz continuous boundary $\Gamma=\partial\Omega$, we consider the stationary Stokes equations

$$-\nabla\cdot\boldsymbol{T}(\boldsymbol{u},p)=\boldsymbol{f}\quad\text{in }\Omega,\tag{1}$$

$$\nabla\cdot\boldsymbol{u}=0\quad\text{in }\Omega,\tag{2}$$

where $\boldsymbol{T}(\boldsymbol{u},p)=-p\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u})$ is the stress tensor, $\boldsymbol{D}(\boldsymbol{u})=(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)/2$ is the deformation rate tensor and $\mu>0$ is the kinematic viscosity of the fluid. On the boundary we prescribe a *slip* boundary condition:

$$\boldsymbol{u}\cdot\boldsymbol{\nu}=g\quad\text{on }\Gamma,\tag{3}$$

$$\boldsymbol{\nu}\cdot\boldsymbol{T}(\boldsymbol{u},p)\cdot\boldsymbol{\tau_i}=f_{2,i},\quad i=1,n-1,\text{ on }\Gamma.\tag{4}$$

Here, $\boldsymbol{\nu}$ is the outgoing unit normal vector to Γ whereas $\boldsymbol{\tau_i}$, $i=1,n-1$, are orthonormal vectors spanning the plane tangent to Γ. When $g=0$ (zero-flux condition) and $f_{2,i}=0$, $i=1,n-1$ (vanishing forces in tangential directions to the boundary), this is also known as the *free slip* boundary condition.

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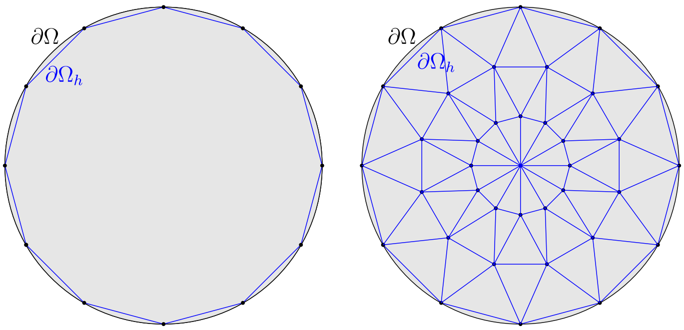

Fig. 1. Smooth domain Ω and its meshed approximation Ω_h.

In the literature, slip-type boundary conditions are less frequent than, for instance, Dirichlet or Neumann (free) boundary conditions. Nevertheless, they are involved in problems with biological surfaces and interfaces [6,7], polymer melts [13], slide coating [11], turbulence models [24], or special problems with Newtonian fluid flows at solid interfaces [25]. They can be viewed as a mix of a Dirichlet boundary condition in the normal direction to the boundary and a Neumann (*i.e. natural*) boundary condition in tangential directions.

In the numerical literature, several weak formulations have been devised to form the basis of a finite element approximation of Stokes or Navier–Stokes equations with a slip boundary condition. The theoretical results, that we review in the following, were established for Ω with a smooth curved (non-polygonal) boundary and finite element approximations were considered for polyhedral domains approximating this smooth domain. These theoretical results differ primarily in the treatment of the boundary flux condition (3).

In one of the simplest formulation, the boundary flux condition (3) is imposed strongly: the velocity approximation is sought in an ansatz space where all vectors satisfy (3) at each nodes when Lagrange finite elements are used. The first convergence results were proved by Verfürth [31]. The convergence rates obtained in [31] are not optimal and were improved by Knobloch [22] and Bänsch and Deckelnick [4], from $1/2$ to $3/2$ in usual norms, in the case of Taylor–Hood (P_2/P_1) elements. This rate cannot be improved further in general due to the error in the approximation of Ω by polygonal domains Ω_h (see for instance Strang and Fix [29, Section 4.4]).

Interestingly, and in direct relation to our present work, in [31] Verfürth argued that the obtained suboptimal rate could be difficult to improve due to Babuska's like paradox [2,3]. Babuska's paradox can be stated as follows: *the solution of Kirchhoff plate equations with simple support boundary conditions in a disk is not the limit of the solutions to the same equations posed on polygonal domains approaching the disk*, as in Fig. 1 (left). One can then easily imagine that some difficulties may appear when performing finite element approximations of these equations in meshed polygonal domains approaching the disk, like in Fig. 1 (right). This paradox holds in fact whenever a smooth curved boundary is involved (see [23]). That Babuska's like paradox is into play in the case of Stokes equations with slip boundary conditions was pointed out by Verfürth [31] by observing that the stream function formulation of Stokes equations with *free* slip boundary conditions, obtained by posing $\boldsymbol{u} = \mathbf{curl}\,\psi$ (which is possible since $\nabla \cdot \boldsymbol{u} = 0$, and then ψ is the so-called *stream function*), leads to the Kirchhoff plate equation $-\mu\Delta^2\psi = \operatorname{curl} \boldsymbol{f}$ with simple support boundary conditions: $\psi = 0$ and $\Delta\psi = 2\kappa\,\partial\psi/\partial\boldsymbol{\nu}$ (where κ is the curvature of $\partial\Omega$, see for instance [12] for details).

Motivated by the lack of optimality in his estimates, Verfürth [32] proposed to handle the constraint Eq. (3) in a weak way, by the Lagrange multiplier method. With the introduction of a new variable – the Lagrange multiplier-finite element approximation spaces need to be chosen with care (enriching the velocity approximation space with bubble functions having their support in the vicinity of Γ, like in [32], for instance) or residual boundary terms may be added in the original variational formulation of the equations, resulting in a so-called *stabilized* formulation [33]. As Stenberg [28] already observed, formal elimination of the Lagrange multiplier in these stabilized formulations results in problem formulations similar to Nitsche's method [26] to weakly append Dirichlet type boundary conditions.

The objective of this work is to study the efficiency of these two methods (the Lagrange multiplier method and Nitsche's method) to impose in a weak way a slip boundary condition (more precisely flux boundary condition (3)) for Stokes (and Navier–Stokes) equations, and to put into evidence some convergence problems which may be related to Babuska's like paradox, in the case of a domain with a smooth curved boundary. As convergence of finite element approximations are difficult to establish in the case of a smooth curved boundary, and as convergence of these methods depends on the values of some parameters (see Sections 2 and 4) even in the case of a polygonal boundary, we first study these methods theoretically only in the case of a polygonal (or polyhedral) boundary. Then we test them numerically, first with a polygonal boundary (in order to illustrate the theoretical results) and then with a smooth boundary in order to put into evidence the paradox.

In Section 2, after recalling the Lagrange multiplier method, we first describe a set of variants of the stabilized formulation introduced and studied in [33]. In Section 3, we prove the convergence of the resulting approximation method when Ω has a polygonal or polyhedral boundary. The theoretical convergence results depend on the values of the parameters. In Section 4, we prove the stability of several variants of Nitsche's method, depending on the values of the parameters, assum-