Journal of Computational Physics 258 (2014) 856–870

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A fast and high-order method for the three-dimensional elastic wave scattering problem

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ARTICLE INFO

Article history:
Received 2 April 2013
Received in revised form 27 October 2013
Accepted 11 November 2013
Available online 18 November 2013

Keywords:
Boundary integral equation method
Elastic wave scattering
FFT

ABSTRACT

In this paper we present a fast and high-order boundary integral equation method for the elastic scattering by three-dimensional large penetrable obstacles. The algorithm extends the method introduced in [5] for the acoustic surface scattering to the fully elastic case. In our algorithm, high-order accuracy is achieved through the use of the partition of unity and a semi-classical treatment of relevant singular integrals. The computational efficiency associated with the nonsingular integrals is attained by the method of equivalent source representations on a Cartesian grid and Fast Fourier Transform (FFT). The resulting algorithm computes one matrix–vector product associated with the discretization of the integral equation with $O(N^{4/3}\log N)$ operations, and it shows algebraic convergence. Several numerical experiments are provided to demonstrate the efficiency of the method.

1. Introduction

Elastic wave scattering plays a significant role in engineering and industrial design and identification. Examples include the recovery of elastic properties of materials and composites, non-destructive testing and geophysical exploration. In this paper, we consider the elastic wave scattering of a time harmonic (with $e^{i\omega t}$ time dependence) incident wave field that impinges on a penetrable obstacle. In particular, we are interested in the case when the size of the obstacle is several dozen times larger than the wavelength of the incident field. Such scattering problems usually exist in seismic exploration, where the domain of interest is usually several kilometers [4], or in elastography for medical imaging, where the size of human tissue is much larger than the wavelength of the high frequency incident beams [10,11]. Accurate numerical solution of such problems is still challenging nowadays, as the wave field is usually highly oscillatory, and in general it requires a large number of grid points to resolve the wave field with sufficient accuracy.

In the past few decades, various advanced numerical simulators have been developed to model the elastic wave scattering, these include the standard finite difference methods (e.g. [17,20,23,27,31,37]), and finite element methods (e.g. [15,16,19,24,32,35]). However, both of them involve discretization over the volume of the obstacle, and their cost even at moderate frequencies quickly becomes prohibitive. Moreover, the infinite exterior domain has to be truncated into a finite one, and an absorbing boundary condition [9] or perfectly matched layer (PML) [3] needs to be imposed on the boundary. Integral equation formulations, on the other hand, offer some advantages from a numerical perspective. Indeed, compared to the finite element or finite difference approaches, the integral equations are only discretized on the surface of the obstacle, which results in a dramatic reduction of computational cost for a given accuracy. Moreover, they enforce the radiation condition automatically, as the very formulation encodes the correct behavior of outgoing waves. We refer the readers

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0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.11.009

to [1,14,21,22,26,28] and references therein for recent developments of boundary element methods for solving the elastic scattering problem.

Although the integral formulations provide significant gains in memory requirements, a brute-force integration scheme would lead to $O(N^2)$ computation cost for each matrix–vector product, where N is the number of grid points. This is still formidable even for supercomputers when N is very large. For this reason, a number of algorithms have been introduced to evaluate the discretized integral equations in a fast way, of which the most celebrated algorithm is the fast multipole methods (FMM) [8,25,29,33,36]. The method provides significant gains in computation time with $O(N\log N)$ operations only. However, it also presents certain limitations, such as low-order accuracy. It is shown that the relative error for the numerical solution may be several percent even for the simplest scatterers (see, for example, [7,34]). Moreover, in the elastic wave scattering, multiple-tree frames are needed due to the existence of both longitudinal and transverse waves with different wave speeds in one medium. This results in a nonuniform definition for well-separated groups and greatly complicates the implementation. We refer the readers to [36] for more details in this regard.

The goal of this paper is to present an accurate and efficient boundary integral equation method to solve elastic scattering by large obstacles. A partition of unity and a semi-classical method is employed to evaluate the singular integrals accurately. Nonsingular integrals are evaluated by high-order quadrature rules with a fast method, wherein each matrix–vector product is evaluated with $O(N^{4/3}\log N)$ operations. Our acceleration strategy is based on the two face equivalent source approximation, which reduces the evaluation of nonadjacent interactions in the integral formulations to an evaluation of 3-D Fast Fourier Transform (FFT). It is a nontrivial extension of the acoustic solver in [5] to allow for the evaluation of the elastic wave scattering in three dimensions. It should be mentioned that additional difficulties arise in the context of the elastic wave scattering, since compared to the scalar acoustic wave scattering, the vector elastic wave field consists of a longitudinal and a transverse part propagating at different speeds. Moreover, due to the essential differences in the singularity of Green's tensors, instead of dealing with integrals with weakly singular kernels only, here we have to evaluate integrals with singular and even hypersingular kernels. The resulting fast high-order method attains the accuracy of traditional boundary element method with a significant lower computational cost, especially when N is large. It is capable of handling a wide variety of complex large obstacles. In addition, the equivalent sources are solved independently within each small subdomain, which makes the algorithm suitable for parallelism.

The rest of the paper is organized as follows. Section 2 introduces the mathematical model for the elastic wave scattering problem, and formulates the boundary integral equations. The numerical method is presented in Section 3, where accurate evaluation of singular integrals and acceleration for the far interactions of integrations are discussed in details. We show various numerical results in Section 4 to demonstrate the efficiency of the method, and conclude with some remarks in Section 5.

2. Problem formulation and boundary integral equations

Let Ω_1 denote the body of the obstacle, and $\Omega_2=\mathbb{R}^3\backslash\bar{\Omega}_1$ be its exterior region. The Lamé constants for the regions inside and outside the obstacle are γ_i and μ_i $(i=1,2)$ respectively, and the densities are ρ_i $(i=1,2)$. Assume that an incident wave field $\mathbf{u}^{inc}$ impinges on the obstacle Ω_1. The displacement of the scattered wave and that of the wave excited inside the obstacle satisfy the following elastic wave equations in the frequency domain

$$\frac{1}{k_{p,i}^2}\nabla\nabla\cdot\mathbf{u}_i-\frac{1}{k_{s,i}^2}\nabla\times\nabla\times\mathbf{u}_i+\mathbf{u}_i=0,\quad x\in\Omega_i,\ i=1,2. \tag{2.1}$$

Here $k_{p,i}=\frac{\omega}{c_{p,i}}$ and $k_{s,i}=\frac{\omega}{c_{s,i}}$ are wavenumbers for the longitudinal wave (P-wave) and the transverse wave (S-wave) inside and outside Ω_1, respectively. ω is the operating frequency of the elastic wave, and the wave speeds are given by

$$c_{p,i}=\sqrt{\frac{\gamma_i+2\mu_i}{\rho_i}}\quad\text{and}\quad c_{s,i}=\sqrt{\frac{\mu_i}{\rho_i}},\quad i=1,2.$$

Across the surface Γ of the obstacle, the displacement $\mathbf{u}$ and the traction field $\mathbf{t}$ are continuous. That is,

$$\mathbf{u}_1=\mathbf{u}_2+\mathbf{u}^{inc},\qquad \mathbf{t}_1=\mathbf{t}_2+\mathbf{t}^{inc}\quad\text{on }\Gamma. \tag{2.2}$$

Here the traction field is given by $\mathbf{t}_i=\gamma(\nabla_y\cdot\mathbf{u}_i(x,y))\mathbb{I}+\mu(\nabla_y\mathbf{u}_i(x,y)+\nabla_y\mathbf{u}_i^T(x,y))\cdot\mathbf{n}(y)$ $(i=1,2)$. $\mathbb{I}$ is the identity matrix, and $\mathbf{n}$ is unit outward normal to the surface Γ. The definition for $\mathbf{t}^{inc}$ follows a similar fashion.

At infinity, the scattered wave $\mathbf{u}_2$ satisfies the Sommerfeld radiation condition. More precisely, the displacement $\mathbf{u}_2$ admits the Helmholtz decomposition

$$\mathbf{u}_2=\nabla\phi+\nabla\times\boldsymbol{\psi}, \tag{2.3}$$

where ϕ and $\boldsymbol{\psi}$ satisfy the following radiation conditions:

$$\lim_{r\to\infty} r\left(\frac{\partial\phi}{\partial r}-\mathrm{i}k_{p,2}\boldsymbol{\phi}\right)=0; \tag{2.4}$$