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Generalized multiscale finite element method. Symmetric interior penalty coupling

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ABSTRACT

Motivated by applications to numerical simulations of flows in highly heterogeneous porous media, we develop multiscale finite element methods for second order elliptic equations. We discuss a multiscale model reduction technique in the framework of the discontinuous Galerkin finite element method. We propose two different finite element spaces on the coarse mesh. The first space is based on a local eigenvalue problem that uses an interior weighted L_2-norm and a boundary weighted L_2-norm for computing the "mass" matrix. The second choice is based on generation of a snapshot space and subsequent selection of a subspace of a reduced dimension. The approximation with these multiscale spaces is based on the discontinuous Galerkin finite element method framework. We investigate the stability and derive error estimates for the methods and further experimentally study their performance on a representative number of numerical examples.

1. Introduction

In this paper we present a study of numerical methods for the simulation of flows in highly heterogeneous porous media. The media properties are assumed to contain multiple scales and high contrast. In this case, solving the systems arising in the approximation of the flow equation on a fine grid that resolves all scales by the finite element, finite volume, or mixed FEM could be prohibitively expensive, unless special care is taken for solving the resulting system. A number of techniques have been proposed to efficiently solve these fine-grid systems. Among these are multigrid methods (e.g., [5,14]), multilevel methods (e.g., [32,33]), and domain decomposition techniques (e.g., [16,18,23–25,31]).

More recently, a new large class of accurate reduced-order methods has been introduced and used in various applications. These include Galerkin multiscale finite elements (e.g., [3,9,13,20–22]), mixed multiscale finite element methods (e.g., [1,2,4,27]), the multiscale finite volume method (see, e.g., [28]), mortar multiscale methods (see, e.g., [6,34]), and variational multiscale methods (see, e.g., [26]). Our main goal is to extend these concepts and to develop a systematic methodology for solving complex multiscale problems with high-contrast and no-scale separation by using discontinuous basis functions.

In this paper, we study the multiscale model reduction techniques within discontinuous Galerkin framework. As the problem is expected to be solved for many input parameters such as source terms, boundary conditions, and spatial heterogeneities, we divide the computation into two stages (following known formalism [8,30,29]): offline and online, where

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our goal in the offline stage is to construct a reduced dimensional multiscale space to be used for rapid computations in the online stage. In the offline stage [19], we generate a snapshot space and propose a local spectral problem that allows selecting dominant modes in the space of snapshots. In the online stage we use the basis functions computed offline to solve the problem for current realization of the parameters (a further spectral selection may be done in the online step in each coarse block). As a result, the basis functions generated by coarse block computations are discontinuous along the coarse-grid inter-element faces/edges. Previously, e.g. [19], in order to generate conforming basis functions, multiplication by partition of unity functions has been used. However, this procedure modifies original spectral basis functions and is found to be difficult to apply for more complex flow problems. In this paper, we propose and explore the use of local model reduction techniques within the framework of the discontinuous Galerkin finite element methods.

We introduce a Symmetric Interior Penalty Discontinuous Galerkin (SIPG) method that uses spectral basis functions that are constructed in special way in order to reduce the degrees of freedom of the local (coarse-grid) approximation spaces. Also we discuss the use of penalty parameter in the SIPG method and derive a stability result for a penalty that scales as the inverse of the fine-scale mesh. We show that the stability constant is independent of the contrast. The latter is important as the problems under consideration have high contrast.

We also derive error estimates and discuss the convergence issues of the method. Additionally, the efficacy of the proposed methods is demonstrated on a set of numerical experiments with flows in high-contrast media where the permeability fields have subregions of high conductivity, which form channels and islands. In both cases we observe that as the dimension of the coarse-grid space increases, the error decreases and the decrease is proportional to the eigenvalue that the corresponding eigenvector is not included in the coarse space. In particular, we present results when the snapshot space consists of local solutions.

The paper is organized in the following way. In Section 2, we present our model problem in a weak form and introduce the approximation method that involves two grids, fine (that resolves all scales of the heterogeneity) and coarse (where the solution will be sought). On each cell of the coarse mesh we introduce a lower dimensional space of functions that are defined on the fine mesh. We also show that the method is stable in a special DG norm. In Section 3, we present two main choices of local spaces. The first one is based on a few eigenfunctions of special spectral problems in the style of [16,23]. The second choice is based on the concept of snapshots [17]. In Section 4, we present some numerical experiments and report the error of the discontinuous Galerkin method with the constructed coarse-grid spaces and in Section 5, we discuss the numerical results. The theoretical results are derived under the assumption that the penalty stabilization depends on the fine-mesh size. Based on the numerical experiments we can conclude that the interior penalty Galerkin method gives reasonable practical results in using coarse-grid spaces generated by special problems, solved locally on each coarse-grid block, that take into account the highly heterogeneous behavior of the coefficient of the differential equation (in our case, the permeability).

2. Continuous and discrete problems

We consider the following problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a\big(u^*, v\big) = f(v) \quad \text{for all } v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u, v) := \int\limits_{\Omega} \kappa(x) \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int\limits_{\Omega} f v \, dx.$$

Here Ω is a bounded domain in R^d, $d = 2, 3$ with polygonal boundary. We assume that $f \in L_2(\Omega)$ and the coefficient $\kappa(x)$ represents the permeability of a highly heterogeneous porous media with high contrast, that is high ratio between the maximum and minimum values, see Fig. 1. Our main goal in this paper is to develop an approximation method for (1) on a coarse grid using certain "low energy" local eigenfunctions within the discontinuous Galerkin framework.

We consider the two dimensional case. The method and results presented here extend for the three dimensional case. We split the domain Ω into disjoint polygonal subregions $\{\Omega_i\}_{i=1}^N$ of diameter $O(H_i)$ so that $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$. We assume that the substructures $\{\Omega_i\}_{i=1}^N$ form a geometrically conforming partition of Ω. In this case, for $i \neq j$, the intersection $\partial\Omega_i \cap \partial\Omega_j$ is either empty, a vertex of Ω_i and/or Ω_j, or a common edge of $\partial\Omega_i$ and $\partial\Omega_j$.

Further, in each Ω_i we introduce a shape regular triangulation $\mathcal{T}_h(\Omega_i)$ with triangular elements and maximum mesh size h_i. The resulting triangulation of Ω is in general nonmatching across $\partial\Omega_i$. Let $X_h(\Omega_i)$ be the regular finite element space of piecewise linear and continuous functions in $\mathcal{T}_h(\Omega_i)$. We do not assume that functions in $X_h(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. We define

$$X_h(\Omega) = X_h(\Omega_1) \times \cdots \times X_h(\Omega_N)$$

and represent functions v of $X_h(\Omega)$ as $v = \{v_i\}_{i=1}^N$ with $v_i \in X_h(\Omega_i)$. For simplicity, we also assume that the permeability $\kappa(x)$ is constant over each fine-grid element.