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# On the remedy against shock anomalies in kinetic schemes

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ABSTRACT

Shock-capturing schemes often exhibit anomalous behaviors, such as the carbuncle phenomenon and the post-shock oscillations, especially in the hypersonic flow regime. This paper proposes a simple and effective remedy against these shock instabilities in the case of the kinetic Lax–Wendroff scheme, where the well-known classic scheme is reinforced by means of an equilibrium distribution function of gas molecules. The pathologies are significantly improved to an acceptable level for practical purposes without any considerable side-effect by locally bringing out the robustness of the equilibrium flux method from the kinetic scheme. The remedy is applied only to the preprocessing of the data at cell-edges. The performance of the fortified kinetic scheme is demonstrated in the problem of a hypersonic inviscid or viscous flow past a blunt body. Comparisons are also made with various advanced shock-capturing schemes at present.



## 1. Introduction

Shock-capturing schemes for the compressible Euler equations have been earnestly studied by plenty of researchers since the 1970s and a lot of advanced schemes are now available. It sounds, nevertheless, a bit too exaggerative to say that the shock-capturing technology presently available is well-matured; even the most advanced schemes at present still exhibit anomalous behaviors which should not be overlooked. The carbuncle phenomenon is one of the representative pathologies and many studies have been devoted to the prevention and the cure against it, see e.g. Refs. [1–12] and the references therein. The occurrence of appreciable distortion behind a strong shock wave and the deformation of the shock wave, which sometimes appears in a crooked shape, are cited as its symptoms. This catastrophic pathology is said to be intrinsic to the compressible Euler equations [3,8]. It is not, however, peculiar to the computation of inviscid flows and physical dissipation often fails to prevent it. For example, the reliability of the CFD prediction of aerodynamic heating of a blunt body flying at a hypersonic speed, such as a spacecraft and a meteorite during atmospheric entry, is considerably deteriorated by the onset of the pathology. Besides, the bow shock in front of the body often generates numerical disturbances, i.e. the post-shock oscillations [13], and they can also affect the downstream boundary-layer, where the temperature gradient is extremely steep. These shock instabilities degrade the quality of computations of various shock-wave phenomena, such as shock-acoustic and shock-vortex interactions, transition to turbulence behind a shock wave, and so on.

The shock-capturing strategy, which was discovered by Godunov more than half a century ago [14], is now adopted in various numerical schemes. When the distribution of a numerical solution is reconstructed from its discrete data, discontinuities are intentionally introduced at cell-edges. A state is determined at each cell-edge as the solution of the corresponding Riemann problem and the numerical flux for the state is accompanied by numerical dissipation, which is effective for the

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suppression of the occurrence of spurious oscillations around shocks. An excess of numerical dissipation, however, smears the numerical solution and its amount is cleverly controlled by a sophisticated method for preprocessing, such as MUSCL, ENO, WENO, and so forth, where high order accuracy with respect to space is retained except around shock waves. Ruminating about what Riemann solvers essentially do, we immediately notice or realize again Riemann solvers as the tools for the production of numerical dissipation. The simple arithmetic mean leads to a classical non-shock-capturing scheme and the nonlinearity in the averaging is of crucial importance. Without a priori knowledge of the explicit numerical dissipation desirable for shock-capturing, good tools for dissipation production are obtained only by empirical approaches. While mathematical theories are available for some simple cases, such as scalar conservation laws and 1st order accurate schemes for one space dimension, practitioners involved in projects of engineering or physics still have to resort to empiricism, in particular when they encounter the abovementioned shock anomalies. All efforts for high order accuracy are reduced to ashes in the presence of shock instabilities. Under such situations, there is no rational reason to hesitate to look for tools other than Riemann-solvers.

From the two elementary facts that (i) the compressible Euler equations describe the dynamical behavior of a gas in a local equilibrium state and (ii) the velocity distribution of gas molecules in an equilibrium state is given by the Maxwell distribution function (Maxwellian), we immediately recognize that Maxwellian yields the flux functions for the compressible Euler equations. This allows us to employ Maxwellian as a gadget for computing the average of two states in shock-capturing schemes. The equilibrium flux method (EFM) of Pullin [15], which is one of the originals of kinetic schemes, employs the same piecewise constant reconstruction as in Godunov's method. The numerical flux at a cell-edge is computed from the discontinuous distribution of gas molecules arriving from the two equilibrium states on both sides of the cell-edge, say the discontinuous Maxwellian. The noteworthy feature of EFM is found in its robustness; EFM is more robust than Godunov's method owing to the numerical dissipation generated by the discontinuous Maxwellian. At the same time EFM is too diffusive and is far from boundary-layer capturing. This drawback is partially resolved in the spatially high order accurate treatment, i.e. the kinetic flux vector splitting scheme (KFVS) [16], where a conventional method for high order accurate preprocessing, such as MUSCL, is introduced and the time variation of the numerical flux is taken into account by using the collisionless Boltzmann equation. Although KFVS (or the second order accurate EFM) can capture shock waves more sharply than the original EFM, however, the time variation of the numerical solution is not consistent with that of the compressible Euler equations up to second order accuracy.

The consistent accuracy with respect to time can be achieved by taking account of a more realistic or physical process with molecular collisions and the gas kinetic BGK scheme (GKB) has been developed along the lines [17,18]. GKB is the first kinetic scheme which combines the capability of shock-capturing and that of boundary-layer-capturing. The latter feature is in contrast to KFVS, which requires a prohibitively large number of mesh points in a boundary-layer as well as a very small time step for its accurate computation [17,19]. The fidelity to the real physics is not, however, the key ingredient of GKB, although it brings high order accuracy in time as the consequence. The key lies in the approximation of the Maxwellian which is employed as the distribution of gas molecules after collisions (the gain term in the BGK approximation). The solution of the BGK equation, on which GKB relies, consists of the initial data and the integral of the gain term along its characteristics (the method of integration factor). The discontinuous Maxwellian is employed in the former and the latter is approximated by using the continuous Maxwellian made from the discontinuous one, which is loyal to the BGK equation. The continuous Maxwellian also generates numerical dissipation but it is less dissipative than the discontinuous one. This immediately makes us recall the analogy to hybrid schemes, such as that of HLLE [21,22] and HLLC [23,24]; the former is dissipative and robust and the latter is less dissipative and works better in smooth regions. In Ref. [20], the classic Lax–Wendroff scheme is reinforced by exploiting these two kinetic averages and the resulting scheme, which will be hereafter called the kinetic Lax–Wendroff scheme (KLW), is shown to be able to cope with GKB.

While the abovementioned efforts have bestowed sophistication upon kinetic schemes, the robustness of the original has been lost; both GKB and KLW are afflicted with shock anomalies. Degeneracy does not, however, necessarily mean the loss of gene. In fact, these two kinetic schemes can vary from a Lax–Wendroff type scheme to EFM depending on the data at cell-edges. It is expected that the robustness of these schemes against shock instabilities is easily fortified without any additive dissipation only by a simple modification of the preprocessing to promote the transformation into EFM around shock waves. In the present study we will establish the authenticity of this prescription. Since the structure of KLW is simpler than that of GKB, we will mainly explain the case of KLW. The rest of the paper proceeds as follows. Section 2 briefly describes the derivation of KLW. Section 3 describes the diagnosis of the flaws of KLW. Section 4 explains the modification of the preprocessing. Its efficacy is confirmed in some preliminary problems. The performance of KLW with the remedy is examined in Section 5. A similar remedy is also applied to GKB and its effect is confirmed there. Some concluding remarks are presented in Section 6, where the essential role of kinetic theory for the robustness of kinetic schemes is discussed. Comparisons are also made with various advanced shock-capturing schemes at present in Sections 4 and 5.

## 2. Kinetic Lax–Wendroff scheme

The derivation of KLW in Ref. [20] does not impose any special prerequisites on the reader. It will be briefly reviewed here in order to keep this paper self-contained. In the following explanation, the 2D compressible Euler equations for a monatomic ideal gas will be considered. The generalization to 3D case can be made straightforwardly. The extension to the case of the Navier–Stokes equations for general ideal gases will be also mentioned.