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Adjoint consistency analysis of residual-based variational multiscale methods ☆

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ABSTRACT

We investigate the conditions under which residual-based variational multiscale methods are adjoint, or dual, consistent for model hyperbolic and elliptic partial differential equations. In particular, while many residual-based variational multiscale stabilizations are adjoint consistent for hyperbolic problems and finite-element spaces, only a few are adjoint consistent for elliptic problems.

1. Introduction

Adjoint variables arise in many applications, including output-based error estimation and adaptation, inverse problems, and optimization. The equations governing the adjoint variables are typically obtained using either the differentiate-then-discretize approach or the discretize-then-differentiate approach [1]. In the former, the adjoint PDE is derived first and then discretized, while in the latter, the primal PDE and functional are discretized and the adjoint linear system follows from differentiation of the algebraic equation.

We say the primal problem is dual, or adjoint, consistent if the discretize-then-differentiate approach leads to a linear system that is equivalent to a consistent discretization of the adjoint PDE. Discretizations that are adjoint consistent have been shown to enjoy certain advantages over discretizations that are adjoint inconsistent.

- Adjoint consistency leads to optimal error estimates in the L^2 norm [2].
- Integral functionals are superconvergent relative to the solution error [3,4]. This is valuable from an efficiency perspective, since, for the same degrees-of-freedom, an adjoint-consistent formulation may have a significantly smaller functional error. The numerical examples presented in Section 5 illustrate this.
- Similarly, functional gradients computed using the adjoint variables are superconvergent. In addition, gradients computed in this fashion are equivalent, within machine precision, to sensitivities computed using algorithmic differentiation [5]. This equivalence is important when using conventional optimization algorithms that expect the user to supply accurate gradients.

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- A posteriori output error estimates based on adjoint-consistent schemes — or schemes that are asymptotically adjoint consistent — exhibit better effectivity [6].

Given these advantages, there has been significant interest in studying the adjoint consistency of various discretizations [2,4,7–10]. In this paper, we investigate the adjoint consistency of variational multiscale methods [11–19]. In particular, we focus on residual-based variational multiscale stabilization (RBVMS) and establish conditions under which this form of stabilization becomes adjoint consistent.

2. Adjoint consistency analysis

We begin by reviewing the concept of adjoint, or dual, consistency [2,4,20]. Consider the following variational problem corresponding to the weak form of a partial differential equation (PDE) on the domain $\Omega \subset \mathbb{R}^d$: find $u \in \mathsf{V}$ such that

$$\mathcal{R}(v, u) = 0, \quad \forall v \in \mathsf{V}, \tag{1}$$

where V denotes an appropriate Hilbert space on the domain Ω, and $\mathcal{R} : \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ is a semilinear form (linear in the first argument). In addition, suppose we are interested in nonlinear integral functionals $\mathcal{J} : \mathsf{V} \to \mathbb{R}$ that depend on the solution to (1); for example, when $\mathcal{R}$ corresponds to the weak form of the Navier–Stokes equations, $\mathcal{J}$ might be the lift or drag force.

To find the dual problem corresponding to the functional $\mathcal{J}$ and primal variational problem (1), we introduce the Lagrangian,

$$\mathcal{L}(\psi, u) \equiv \mathcal{J}(u) + \mathcal{R}(\psi, u), \tag{2}$$

where $\psi \in \mathsf{V}$. We arrive at the dual problem by finding ψ such that variations in $\mathcal{L}$ about u vanish:

$$\delta\mathcal{L} \equiv \mathcal{L}'[u](\psi, \delta u) = \mathcal{J}'[u](\delta u) + \mathcal{R}'[u](\psi, \delta u) = 0, \quad \forall \delta u \in \mathsf{V},$$

or, equivalently

$$\mathcal{J}'[u](v) + \mathcal{R}'[u](\psi, v) = 0, \quad \forall v \in \mathsf{V}, \tag{3}$$

where the prime indicates (Fréchet) linearization with respect to the term in square brackets.

We now turn to the finite-dimensional case and introduce V^h, a space that approximates V. Then, the discretization of the primal variational equation (1) leads to the following problem: find $u^h \in \mathsf{V}^h$ such that

$$\mathcal{R}_h\big(v^h, u^h\big) = 0, \quad \forall v^h \in \mathsf{V}^h. \tag{4}$$

The subscript h on $\mathcal{R}_h$ indicates that operators in the discretized semilinear form may depend on the discrete solution u^h in the nonlinear case. The discretized functional will be denoted by $\mathcal{J}_h : \mathsf{V}^h \to \mathbb{R}$. Note that in general, $\mathcal{R}_h$ and $\mathcal{R}$, and $\mathcal{J}_h$ and $\mathcal{J}$, are not the same.

The analysis used to derive the adjoint variational problem can be applied to the finite-dimensional case. This leads to the following linear system that is satisfied by the discrete adjoint variable $\psi^h \in \mathsf{V}^h$:

$$\mathcal{J}_h'\big[u^h\big]\big(v^h\big) + \mathcal{R}_h'\big[u^h\big]\big(\psi^h, v^h\big) = 0, \quad \forall v^h \in \mathsf{V}^h. \tag{5}$$

Adjoint consistency arises from the relationship between the discrete adjoint variational equation (5) and the infinite-dimensional adjoint solution ψ. For completeness, we include the definition of adjoint consistency below; see, for example, [4].

Definition 1 *(Dual/adjoint consistency).* The finite element discretization (4) and discrete functional $\mathcal{J}_h$ are dual/adjoint consistent if

$$\mathcal{J}_h'[u]\big(v^h\big) + \mathcal{R}_h'[u]\big(\psi, v^h\big) = 0, \quad \forall v^h \in \mathsf{V}^h, \tag{6}$$

where u and ψ are weak solutions to the primal, (1), and adjoint, (3), variational equations, respectively.

In the absence of boundary conditions and stabilization, it is easy to see that a Galerkin finite-element discretization is dual consistent. Adjoint consistency is less clear when stabilization is present. For example, is dual consistency preserved when a residual-based variational multiscale stabilization is introduced in (4)? This is the question we now consider for first- and second-order PDEs.