Journal of Computational Physics 253 (2013) 138–156

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

On the asymptotic preserving property of the unified gas kinetic scheme for the diffusion limit of linear kinetic models

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ARTICLE INFO

Article history:
Received 14 January 2013
Received in revised form 6 June 2013
Accepted 3 July 2013
Available online 12 July 2013

Keywords:
Transport equations
Diffusion limit
Asymptotic preserving schemes
Stiff terms

ABSTRACT

The unified gas kinetic scheme (UGKS) of K. Xu et al. (2010) [37], originally developed for multiscale gas dynamics problems, is applied in this paper to a linear kinetic model of radiative transfer theory. While such problems exhibit purely diffusive behavior in the optically thick (or small Knudsen) regime, we prove that UGKS is still asymptotic preserving (AP) in this regime, but for the free transport regime as well. Moreover, this scheme is modified to include a time implicit discretization of the limit diffusion equation, and to correctly capture the solution in case of boundary layers. Contrary to many AP schemes, this method is based on a standard finite volume approach, it does neither use any decomposition of the solution, nor staggered grids. Several numerical tests demonstrate the properties of the scheme.

1. Introduction

Kinetic models are efficient tools to describe the dynamics of systems of particles, like in rarefied gas dynamics (RGD), neutron transport, semi-conductors, or radiative transfer. Numerical simulations based on these models require important computational resources, but modern computers make it possible to simulate realistic problems.

These simulations can be made much faster when the ratio between the mean free path of particles and a characteristic macroscopic length (the so-called Knudsen number in RGD, denoted by ε in this paper) is small. In such cases, the system of particles is accurately described by a macroscopic model (Euler or Navier–Stokes equations in RGD, diffusion equations in neutron or photon transport) that can be numerically solved much faster than with kinetic models.

However, there are still important problems in which the numerical simulation is difficult: in multiscale problems, ε can be very small in some zones, and very large elsewhere (opaque vs. transparent regions in radiative transfer). Standard numerical methods for kinetic equations are very expensive in such cases, since, for stability and accuracy reasons, they must resolve the smallest microscopic scale, which is computationally expensive in small ε zones. By contrast, macroscopic solvers are faster but may be inaccurate in large ε zones.

This is why multiscale numerical methods have been presented in the past 20 years: the asymptotic preserving (AP) schemes. Such schemes are uniformly stable with respect to ε (thus their computational complexity does not depend on ε), and are consistent with the macroscopic model when ε goes to 0 (the limit of the scheme is a scheme for the macroscopic model).

AP schemes have first been studied (for steady problems) in neutron transport by Larsen, Morel and Miller [26], Larsen and Morel [25], and then by Jin and Levermore [12,13]. For unstationary problems, the difficulty is the time stiffness due

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0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.07.002

to the collision operator. To avoid the use of expensive fully implicit schemes, several semi-implicit time discretizations schemes, based on a decomposition of the distribution function between an equilibrium part and its deviation, have been proposed by Klar [19], and Jin, Pareschi and Toscani [18] (see preliminary works in [17,11] and extensions in [16,15,32,20,21]). Similar ideas have also been used by Buet et al. in [2], Klar and Schmeiser [22], Lemou and Mieussens [29,1], and Carrillo et al. [4,5]. The theory of well balanced schemes is another way to obtain AP schemes, as in the work of Gosse and Toscani [8,9]. Other approaches have been recently proposed by Lafitte and Samaey [24] and Gosse [7], but there extensions to more complex cases is not clear. Finally, the idea of [14] has been renewed to obtain an AP scheme for linear equations on two-dimensional unstructured meshes in the work of Buet, Després, and Franck [3]. All these methods have advantages and drawbacks, and there is still a need for other AP schemes.

A rather different approach has recently been proposed by K. Xu and his collaborators, in the context of rarefied gas dynamics [37]. This method is called unified gas kinetic scheme (UGKS) and is based on a gas kinetic scheme which has been developed by K. Xu since 2000 (see [36] for the first reference and many other references in [37]). Roughly speaking, the UGKS is based on a finite volume approach in which the numerical fluxes contain information from the collision operator. In some sense, it has some connexions with the well balanced schemes developed for hyperbolic problems with source terms in [14,8,9], even if the construction is completely different. While this approach to design AP schemes looks very promising, it seems that it has not yet received the attention it deserves from the kinetic community. This is probably due to the fact that the nice properties of the UGKS presented in [37] are difficult to understand for people who are not specialist of gas kinetic schemes.

However, we believe that the UGKS approach is very general and can benefit to many different kinetic problems. Let us mention that the big advantage of the UGKS with respect to other methods is that is does not require any decomposition of the distribution function (hence there is no problem of approximation of the boundary conditions), it does not use staggered grids (which is simpler for multidimensional problems), and it is a finite volume method (there is no need of discontinuous Galerkin schemes that are more expensive).

In this paper, our first goal is to present UGKS in a very simple framework, so that it can be understood by any researcher interested in numerical method for kinetic equations. We also want to show that the UGKS can be successfully applied to other fields than RGD. Here, it is used to design an AP scheme for linear kinetic equations, namely a simple model of radiative transfer. Such an extension is not obvious, since linear models exhibit a purely diffusive (parabolic) behavior in the small ε regimes, while models from RGD (like the Boltzmann equations) have a rather convection (hyperbolic) behavior. Indeed, even if the UGKS is originally made to correctly describe this convection regime and to capture the small viscous effects (like in the compressible Navier–Stokes equations), we prove in this paper that it can also capture a purely diffusive effect. Moreover, we propose several extensions: implicit diffusion, correct boundary conditions for boundary layers, treatment of collision operator with non-isotropic scattering kernel. The scheme is proved to be AP in both free transport and diffusion regimes, and is validated with several numerical tests.

The outline of our paper is the following. In Section 2, we present the linear kinetic model, and its approximation by the UGKS. Its asymptotic properties are analyzed in Section 3. Some extensions are given in Section 4, and the scheme is validated with various numerical tests in Section 5.

2. The UGKS for a linear transport model

2.1. A linear transport model and its diffusion limit

The linear transport equation is a model for the evolution (by transport and interaction) of particles in some medium. In this paper, we are mainly concerned by the radiative transfer equation, which reads

$$\frac{1}{c}\partial_t\phi + \Omega\cdot\nabla_r\phi = \sigma\left(\frac{1}{4\pi}\int\phi\,d\Omega - \phi\right) - \alpha\phi + G,$$

where $\phi(t,r,\Omega)$ is the spectral intensity in the position–direction phase space that depends on time t, position $r = (x,y,z)\in\mathbb{R}^3$, and angular direction of propagation of particles $\Omega\in S^2$, while c is their velocity (the speed of light). Moreover, σ is the scattering cross section, α is the absorption cross section, and G is an internal source of particles. These three last quantities may depend on x, but they are independent of Ω. The linear operator $\phi\mapsto\frac{1}{4\pi}\int\phi\,d\Omega-\phi$ models the scattering of the particles by the medium and acts only on the angular dependence of ϕ. This simple model does not allow for particles of possibly different energy (or frequency); it is called "one-group" or "monoenergetic" equation.

In order to study the diffusion regime corresponding to this equation, a standard dimensional analysis is made (see [26] for details). We choose a macroscopic length scale r^*, like the size of the computational domain. We assume that this length is much larger than the typical mean free path λ^* (defined by a typical value of $1/\sigma$), and we denote by ε the ratio λ^*/r^* which is supposed to be much smaller than 1. We choose a macroscopic time scale t^* which is much larger than the typical mean free time $\tau^*=\lambda^*/c$, so that $\frac{\tau^*}{t^*}=\varepsilon^2$. Finally, we assume that the absorption cross section α and the source G are of the order $O(\varepsilon^2)$ as compared to σ. Then, with the non-dimensional scaled variables $t'=t/t^*$, $r'=r/r^*$, $\sigma'=\sigma/\sigma^*$, $\alpha'=\alpha/\alpha^*$, $G'=G/G^*$, we get the following equation

$$\varepsilon\partial_{t'}\phi + \Omega\cdot\nabla_{r'}\phi = \frac{\sigma'}{\varepsilon}\left(\frac{1}{4\pi}\int\phi\,d\Omega - \phi\right) - \varepsilon\alpha'\phi + \varepsilon G'.$$