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Computation of three-dimensional standing water waves

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ABSTRACT

We develop a method for computing three-dimensional gravity-driven water waves, which we use to search for time-periodic standing wave solutions. We simulate an inviscid, irrotational, incompressible fluid bounded below by a flat wall, and above by an evolving free surface. The computations make use of spectral derivatives on the surface, but also require computing a velocity potential in the bulk, which we carry out using a finite element method with fourth-order elements that are curved to match the free surface. This computationally expensive step is solved using a parallel multigrid algorithm, which is discussed in detail. Time-periodic solutions are searched for using a previously developed overdetermined shooting method. Several families of large-amplitude three-dimensional standing waves are found in both shallow and deep regimes, and their physical characteristics are examined and compared to previously known two-dimensional solutions.

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1. Introduction

Gravity-driven water waves have been studied for well over a century and have a rich mathematical structure arising from their nonlinearity. Effects such as resonances [1,2] can have important consequences in ocean engineering, and may be exploited in the design of maritime structures [3]. One method of investigating the properties of water waves has been to search for special solutions of the free-surface Euler equations for an inviscid, incompressible fluid such as traveling and standing waves. One of the earliest examples of this is due to Stokes, who in 1880 postulated that the traveling wave of maximum height has a crest with an internal angle of 120°. This was later investigated numerically [4,5], and proved analytically [6]. The self-similar asymptotics of the almost-highest traveling wave has also been investigated [7–9].

A similar proposition was given for standing waves by Penney and Price in 1952 [10]. By considering several terms in a perturbation expansion, they proposed that the largest amplitude standing wave would form a sharp crest with an internal angle of 90°. This prediction was in reasonable agreement with experiments carried out by Taylor [11]. However, subsequent analytical and numerical studies have reached a variety of different conclusions concerning the precise form of the geometric singularity the limiting extreme wave should possess [12–19]. Recently, Wilkening [20] has shown that at higher resolutions, the self-similar sharpening of the crest eventually breaks down, and several families of time-periodic solutions can be found featuring small-scale oscillations near the crest. This casts doubt on the assumption that a limiting wave profile exists at all, much less one with 90° crests.

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All of the aforementioned studies consider two-dimensional (2D) fluids with one-dimensional surfaces, and it is natural to ask how these results may generalize to three dimensions. However, the investigation of three-dimensional standing waves has been comparatively limited. Verma and Keller [21] and Bridges [22] carried out calculations of small-amplitude waves using perturbation expansions. They were able to examine bifurcations in the families of solutions, and to determine how the periods of standing waves vary as a function of amplitude. More recently Bryant and Stiassnie [23] and Zhu et al. [24] have investigated questions of three-dimensional wave stability and evolution while Engsig-Karup et al. [25] have developed a large-scale parallel code for solving the nonlinear evolution of free-surface waves using a finite difference framework. None of these works attempt a computation of large-amplitude three-dimensional standing waves.

Searching for three-dimensional standing water waves offers a number of technical challenges. Some methods used to search for two-dimensional standing waves, such as conformal mapping methods [13,26–29], do not have a generalization to three dimensions. Furthermore, calculating three-dimensional standing waves requires significantly more computational resources. Simulating the wave itself requires one more dimension, and the number of degrees of freedom parameterizing the configuration space over which to search is also larger.

In this paper, we take advantage of improvements in computational power and new algorithms to calculate time-periodic, three-dimensional gravity-driven waves in an incompressible, inviscid fluid. To search for time-periodic solutions, we make use of a recently developed methodology where the problem is framed as an overdetermined nonlinear system and a minimization technique is employed to search a configuration space for solutions that are progressively closer to being time-periodic. This approach has been used to find time-periodic solutions of the Benjamin–Ono equation [30,31], the vortex sheet with surface tension [32], and two-dimensional standing water waves [33]. Different minimization methods have been employed, such as an adjoint-based BFGS approach [34], but here we make use of a variant of the trust-region based Levenberg–Marquardt minimization. This technique requires computing an entire Jacobian: at a given wave configuration, it is necessary to determine how the time-periodicity will change in each direction of the configuration space. While this is expensive to calculate, the minimization requires far fewer iterations than the BFGS approach, and is more amenable to parallelization. Newton–Krylov methods [35,36] would be an interesting alternative to explore; however, these methods generally work better in externally driven, dissipative systems [37–39] in which viscosity damps high-frequency oscillations as the solution evolves.

Computation of irrotational water waves requires time-integrating the height of the free surface and a velocity potential on the surface. However, at each step, it is also necessary to solve the Laplace equation in the three-dimensional bulk of the fluid. We have developed a fourth-order finite element discretization of the fluid domain for this purpose, where the order is measured with respect to the H^1 Sobolev norm. The use of finite elements is not typical, and for two-dimensional studies, boundary integral methods are more common. However in three dimensions, an argument can be made that a finite element discretization is more suitable, since it requires solving an $O(N^3)$ sparse linear system, as opposed to an $O(N^2)$ dense linear system for a boundary integral approach (where N is a typical number of grid points in one dimension). This difference in complexity is more favorable in two dimensions, where the comparison would be between an $O(N^2)$ sparse system and an $O(N)$ dense system. Naturally, fast algorithms can be used to reduce the computational cost of the boundary integral approach, but the prefactors are currently very large in three-dimensional implementations of these algorithms [40].

Solving the finite element problem is the most computationally intensive part of our fluid solver. To carry this out, we have developed a parallel geometric multigrid algorithm, which is presented in detail below. The multigrid algorithm can also compute solutions for several right-hand sides concurrently, and due to memory bandwidth considerations, this can be carried out in a fraction of the time required to compute each solution sequentially. This feature is exploited in the computation of the Jacobian needed in the Levenberg–Marquardt minimization.

In this paper, we present several families of time-periodic solutions that we have found using this methodology. We examine waves in two depth regimes, one relatively shallow (with fluid depth equal to 1/12 the wavelength) and one relatively deep (with fluid depth equal to 1/2 the wavelength). The distinction boils down to whether $\tanh kh$ is close to 1, where k is the wave number and h is the depth. Given the difficulties of computation, whereby calculating a single time-periodic solution can take several days using sixteen threads, the numerical results we present are of relatively low resolution when compared to two-dimensional studies, and we exploit a large amount of symmetry in order to reduce the dimension of the configuration space that must be searched. Since little is known about three-dimensional standing waves, our main aim in this paper is to examine their physical characteristics and compare them to two-dimensional solutions, paying particular attention to ways in which there may be significant differences. In three dimensions, we are also able to ask fundamentally new questions about wave morphology that would not be applicable in two dimensions. Our results cover only a very small part of the possible range of three-dimensional time-periodic solutions that may exist, but serve to highlight some interesting questions for further study.

2. Methods

2.1. Governing equations

We make use of an (x, y, z) coordinate system that is periodic in the horizontal x and y directions, and where gravity g points in the negative z direction. We employ non-dimensionalized units, where $g = 1$ and the horizontal coordinates cover the range $[0, 2\pi)$. The fluid is bounded below by a flat base at $z = 0$, and has a free surface given by $z = \eta(x, y, t)$. The