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Genuinely multi-dimensional explicit and implicit generalized Shapiro filters for weather forecasting, computational fluid dynamics and aeroacoustics

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ABSTRACT

This paper addresses the extension of one-dimensional filters in two and three space dimensions. A new multi-dimensional extension is proposed for explicit and implicit generalized Shapiro filters. We introduce a definition of explicit and implicit generalized Shapiro filters that leads to very simple formulas for the analyses in two and three space dimensions. We show that many filters used for weather forecasting, high-order aerodynamic and aeroacoustic computations match the proposed definition. Consequently the new multi-dimensional extension can be easily implemented in existing solvers. The new multi-dimensional extension and the two commonly used methods are compared in terms of compactness, robustness, accuracy and computational cost. Benefits of the genuinely multi-dimensional extension are assessed for various computations using the compressible Euler equations.

1. Introduction

Spatial filters are used in weather forecasting, computational fluid dynamics and computational aeroacoustics. These filters can be used to smooth initial conditions and computed solutions or to ensure stability and convergence of the computation. In both cases, these filters are designed to cancel spurious waves, or 2Δ waves, due to errors and uncertainties in measurements or due to rounding errors and numerical instabilities in computations.

The use of spatial filtering for the acceleration of the convergence of computations in meteorological modeling was introduced in the early 1950's by Fjørtoft [1,2] who used a multi-dimensional smoothing operator based on a Laplacian. In the same research field and quite at the same time, Shuman [3] proposed the use of one-dimensional smoothing operators to filter out short-wavelength components of the numerical meteorological fields. Since weather forecasting is by essence multi-dimensional, the different ways to apply these one-dimensional filters in the multi-dimensional case were early discussed in details [3]. A major improvement in the design of spatial filters is due to Shapiro [4–6], who introduced in the 1970's a class of explicit linear filters of the $2N$ order accuracy, using a stencil of $(2N+1)$ points. Shapiro filters ensure the full damping of the shortest wavelengths while totally preserving the largest structures. Again, the different ways to apply these filters in the multi-dimensional case have been already discussed at that time [5]. Shapiro's work was followed by

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many others providing a wide range of linear filters: some intrinsically multi-dimensional [7], some with a variable cut-off frequency [8] and many implicit in space [9–13], *i.e.* using Padé fractions. The latter require the solution of linear algebra but allow controlling the filter stiffness and the location of the cut-off frequency. A detailed review on the use of spatial filters for numerical weather prediction is presented in Ref. [14].

For compressible aerodynamics, the Shuman filter has been employed in the late 1960's to damp the oscillations that were not prevented by centered dissipative schemes near boundaries and shocks [15–17]. The use of this second-order linear filter led to numerical shock structures without spurious oscillations but also very smooth. The stiffness of the shock wave would have been preserved by higher order Shapiro filters, but since they are linear operators, Gibbs phenomenon would result in grid to grid oscillations near the shock. However these oscillations ceased to be a problem thanks to the use of second-order non-centered schemes [18,19] leading to accurate flow solutions with non-oscillatory shock structures. Consequently, spatial filters were no longer employed for shock computations until dedicated nonlinear filters were introduced [20]. A renewed interest in linear filters as artificial dissipation occurred in the 1990's after Lele [21] introduced them in conjunction with high-order centered schemes. This approach has been followed by Gaitonde et al. [22,23] who introduced implicit filters with a variable cut-off frequency by adding a free parameter. Also in the 1990's, Tam et al. [24] proposed low-order numerical schemes and filters for aeroacoustic computations that were optimized in terms of spectral response rather than formal accuracy. This approach was pursued by Bogey and Bailly [25] who derived a family of spatial filters optimized in the Fourier space for aeroacoustic computations. While spatial filters for aerodynamic and aeroacoustic computations are mostly applied to multi-dimensional problems, their design and theoretical study are almost always performed in the one-dimensional case. This approach allows using the same formalism to present both the numerical scheme and the filter but it conceals that the dissipation properties of spatial filters used in two and three space dimensions depend strongly on their numerical implementation. More, one-dimensional spatial filters become anti-dissipative if the wrong multi-dimensional implementation is chosen [26].

This paper addresses the extension of one-dimensional filters in two and three space dimensions. The analysis and developments presented in the following apply to all the linear filters mentioned above. The method used to address all these filters at once rely on a projection of the filters on an orthonormal basis with independent vectors being the Shapiro filters. As will be shown, many filters can be written as linear combinations of Shapiro filters. We will refer to these filters as generalized Shapiro filters and show that they allow the generalization of the analysis and developments to any accuracy order and space dimensions while keeping very simple formulas. The paper is organized as follows. Section 2 introduces the formalism used in the paper, recalls the basics of Shapiro filters and specifies the notion of generalized Shapiro filters. Generalized Shapiro filter decompositions of well known explicit and implicit filters are explicated and discussed. Section 3 presents the two common methods used to extend one-dimensional filters in two and three space dimensions. The pros and cons of each method are discussed. Section 4 describes the design principles and properties of a new, genuinely multi-dimensional, extension of one-dimensional filters. Generic formulas are easily derived for explicit and implicit generalized Shapiro filters. Examples of application of the multi-dimensional extension are given for some filters cited above. In Section 5, the genuinely multi-dimensional filters are applied, in conjunction with non-dissipative high-order schemes, to the computation of aerodynamic, aeroacoustic and atmospheric test cases: the numerical evolution of a stationary vortex, the advection of a vortex, a shock–vortex interaction and a bubble convection. The advantages and drawbacks of the new multi-dimensional filters are discussed in terms of compactness, robustness, accuracy and computational cost. Conclusions are finally drawn on the benefit of the genuinely multi-dimensional filters.

2. Generalized Shapiro filters

2.1. Mathematical formalism

Let w_i be a mesh function defined on a uniform grid ($x_i = i\Delta x$) with spatial step Δx. Applying a spatial filter F to variable w provides

$$\tilde{w}_i = F(w_i) \tag{1}$$

where $\tilde{w}_i$ is the filtered value of the variable w at point $x_i = i\Delta x$ resulting from the application of the filter F. We introduce the standard difference operator [27]:

$$\delta w_i = w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}} \tag{2}$$

that will be extensively used throughout this paper. Since we consider filtering a spatial field known only at points of indices i with integer values, only even compositions of the δ operator will be used in the following. The coefficients applied to the discrete values of w are easily obtained since for a $2n$ composition of the δ operator, with n a positive integer, they are given by the $(2n+1)$ line of Pascal's arithmetic triangle corresponding to the development of $(a-b)^n$, for instance:

$$\begin{aligned} \delta^2 w_i &= w_{i+1} - 2w_i + w_{i-1} \\ \delta^4 w_i &= w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2} \end{aligned} \tag{3}$$