Journal of Computational Physics 252 (2013) 142–162

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Spectral properties of high-order residual-based compact schemes for unsteady compressible flows

K. Grimich, P. Cinnella *, A. Lerat

DynFluid Lab., Arts et Metiers ParisTech, 151 Boulevard de l'Hopital, 75013 Paris, France

ARTICLE INFO

Article history:
Received 25 July 2012
Received in revised form 9 April 2013
Accepted 3 June 2013
Available online 27 June 2013

Keywords:
Residual-based scheme
High order
Unsteady Euler equations
Spectral analysis
Numerical dissipation

ABSTRACT

The wave propagation (spectral) properties of high-order Residual-Based Compact (RBC) discretizations are analyzed to obtain information on the evolution of the Fourier modes supported on a grid of finite size. For these genuinely multidimensional and intrinsically dissipative schemes, a suitable procedure is used to identify the modified wave number associated to their spatial discretization operator, and their dispersive and dissipative behaviors are investigated as functions of a multidimensional wave number. For RBC schemes of higher orders (5 and 7), both dissipation and dispersion errors take very low values up to reduced wave numbers close to the grid resolvability limit, while higher frequencies are efficiently damped out. Thanks to their genuinely multidimensional formulation, RBC schemes conserve good dissipation and dispersion properties even for flow modes that are not aligned with the computational grid. Numerical tests support the theoretical results. Specifically, the study of a complex nonlinear problem dominated by energy transfer from large to small flow scales, the inviscid Taylor–Green vortex flow, confirms numerically the interest of a well-designed RBC dissipation to resolve accurately fine scale flow structures.

1. Introduction

Residual-Based Compact (RBC) schemes have been developed from some time now [1–4] for computing multidimensional, inviscid and viscous, steady and unsteady, compressible flows. Differently from standard numerical schemes that approximate space derivatives independently in each space direction, RBC schemes seek for a compact approximation of the complete residual r, i.e. the sum of all derivatives in the governing equations. Because of this feature, RBC schemes belong to the group of so-called genuinely multidimensional schemes such as the fluctuation splitting schemes [5,6] or the Residual Distribution schemes [7,8]. This makes a deep difference with respect to other compact schemes [9–12] based on the recursive inversion of Pade operators per each space dimension. Moreover, RBC schemes contain a well-designed intrinsic dissipation, also built on derivatives of the residual r, that becomes high-order accurate as r tends to 0. In the recent work [13], a truncation error analysis has been carried out in the nonlinear multidimensional case to identify the high-order dissipation operator associated to a general RBC scheme and to obtain a necessary and sufficient condition (referred to as the χ-criterion) ensuring dissipation for any 2D or 3D situation. Numerical experiments [1–4,13,14] show that dissipative RBC schemes capture flow discontinuities without need for flux limiters or artificial viscosity.

The aim of the present paper is to move a further step toward the understanding of the numerical properties of high-order RBC discretizations, and precisely of the internal representation of solution modes provided by these schemes. Truncation error analysis provides information about the asymptotic behavior of numerical schemes in the limit of vanishing

* Corresponding author.
E-mail address: paola.cinnella@ensam.eu (P. Cinnella).

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.06.005

mesh size. Namely, for stable methods and for smooth flow problems, it allows to conclude on the convergence rate of the global error. Furthermore, it may be used to establish, as done in [13], the dominant (dissipative or dispersive) nature of the numerical scheme, according to the kind of derivatives (even or odd) of the unknown field featuring in the leading error term. Nevertheless, this type of analysis does not provide all possible information on the actual error introduced by the scheme on finite computational grids, and precisely on the cutoff frequencies associated to the numerical representation of the solution. For this purpose, the wave propagation (spectral) properties of the scheme can be studied to obtain information on the evolution of the Fourier modes of the computed field that are supported on a given grid of finite size. The spectral behavior of high-order schemes has been extensively investigated in the past, namely in view of their application to aeroacoustics [15,16] and Large-Eddy Simulation [16,17]. Specifically, a careful analysis of the approximation of convective terms in the governing equations is in order for the numerical simulation of high-Reynolds compressible flows, since it is likely to introduce dispersion and diffusion errors that affect the numerical representation of a given solution mode. For directional schemes, spectral analysis is often applied to a single space derivative taken apart [15–17]. Moreover, since many high-order schemes have a purely centered nature, only the dispersion errors are taken into account, numerical damping being introduced *a posteriori* via the addition of some form of artificial dissipation or explicit filtering, whose transfer function (rate of damping associated to a given wave number) is investigated separately.

This kind of disjoint analysis is not applicable to RBC schemes because of their genuinely multidimensional and intrinsically dissipative nature. For these schemes, an analysis of the properties of the multidimensional spatial discretization operator as a whole is required. In this case, the dispersive and dissipative behavior depends on a multidimensional wave number (or on the local advection direction). Since our main goal is to investigate the spectral properties of numerical approximations for the convective terms, in the following we restrict our analysis to inviscid compressible flow problems.

The paper is organized as follows. In Section 2 we briefly recall the general design principle of RBC spatial discretization for convective problems, then we focus on selected RBC schemes of third-, fifth- and seventh-order of accuracy. In Section 3 we derive the spectral counterparts of the RBC schemes under investigation and discuss their dissipation and dispersion properties. Section 4 presents some numerical experiments supporting the preceding theoretical analysis, including the highly nonlinear Taylor–Green vortex flow [18]. Finally, conclusions of the study are drawn in Section 5.

2. High-order RBC schemes

In this section, we recall the design principles of RBC approximations of the space derivatives for a hyperbolic system of conservation laws. For brevity and clarity, we will focus on two-dimensional problems even if there is no restriction for extending the analyses below to multidimensional hyperbolic problems. At this stage, we treat time derivatives exactly, i.e. we focus on semi-discrete approximations in space.

2.1. Concept of residual-based scheme

Let us consider an initial-value problem for the hyperbolic system of conservation laws:

$$w_t + f_x + g_y = 0 \quad \text{on } \mathbb{R}^2 \times \mathbb{R}^+ \tag{1}$$

with initial conditions

$$w(x, y, 0) = w_0(x, y)$$

where t is the time, x and y are Cartesian space coordinates, w is the state vector and $f = f(w)$, $g = g(w)$ are flux components depending smoothly on w. The Jacobian matrices of the flux are denoted $A = \mathrm{d}f/\mathrm{d}w$ and $B = \mathrm{d}g/\mathrm{d}w$. System (1) is approximated in space on a uniform mesh ($x_j = j\delta x$, $y_k = k\delta y$) with steps δx and δy of the same order of magnitude, say $\mathcal{O}(h)$, using the basic difference and average operators:

$$(\delta_1 v)_{j+\frac{1}{2},k} = v_{j+1,k} - v_{j,k}, \qquad (\delta_2 v)_{j,k+\frac{1}{2}} = v_{j,k+1} - v_{j,k},$$

$$(\mu_1 v)_{j+\frac{1}{2},k} = \frac{1}{2}(v_{j+1,k} + v_{j,k}), \qquad (\mu_2 v)_{j,k+\frac{1}{2}} = \frac{1}{2}(v_{j,k+1} + v_{j,k})$$

where j and k are integers or half integers.

A *residual-based* scheme can be expressed in terms of approximations of the exact residual:

$$r := w_t + f_x + g_y. \tag{2}$$

More precisely, such a scheme is of the following form:

$$(\tilde{r}_0)_{j,k} = \tilde{d}_{j,k} \tag{3}$$

where $\tilde{r}_0$ is a space-centered approximation of r called the *main residual* and $\tilde{d}$ is a residual-based dissipation term defined in terms of first-order differences of the residual as: