Journal of Computational Physics 252 (2013) 495–517

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Fractional Sturm–Liouville eigen-problems: Theory and numerical approximation

Mohsen Zayernouri, George Em Karniadakis *

Division of Applied Mathematics, Brown University, 182 George, Providence, RI 02912, USA

ARTICLE INFO

Article history:
Received 11 March 2013
Received in revised form 25 June 2013
Accepted 27 June 2013
Available online 4 July 2013

Keywords:
Regular/singular fractional Sturm–Liouville operators
Poly-fractonomial eigenfunctions
Fractal expansion set
Exponential convergence

ABSTRACT

We first consider a regular fractional Sturm–Liouville problem of two kinds RFSLP-I and RFSLP-II of order $\nu \in (0,2)$. The corresponding fractional differential operators in these problems are both of Riemann–Liouville and Caputo type, of the same fractional order $\mu = \nu/2 \in (0,1)$. We obtain the analytical eigensolutions to RFSLP-I & -II as non-polynomial functions, which we define as Jacobi *poly-fractonomials*. These eigenfunctions are orthogonal with respect to the weight function associated with RFSLP-I & -II. Subsequently, we extend the fractional operators to a new family of singular fractional Sturm–Liouville problems of two kinds, SFSLP-I and SFSLP-II. We show that the primary regular boundary-value problems RFSLP-I & -II are indeed asymptotic cases for the singular counterparts SFSLP-I & -II. Furthermore, we prove that the eigenvalues of the singular problems are real-valued and the corresponding eigenfunctions are orthogonal. In addition, we obtain the eigen-solutions to SFSLP-I & -II analytically, also as non-polynomial functions, hence completing the whole family of the Jacobi poly-fractonomials. In numerical examples, we employ the new poly-fractonomial bases to demonstrate the exponential convergence of the approximation in agreement with the theoretical results.

1. Introduction

The Sturm–Liouville theory has been the keystone for the development of spectral methods and the theory of self-adjoint operators [1]. For many applications, the Sturm–Liouville Problems (SLPs) are studied as boundary-value problems [2]. However, to date mostly integer-order differential operators in SLPs have been used, and such operators do not include any fractional differential operators. Fractional calculus is a theory which unifies and generalizes the notions of integer-order differentiation and integration to any real- or complex-order [3–5].

Over the last decade, it has been demonstrated that many systems in science and engineering can be modeled more accurately by employing fractional-order rather than integer-order derivatives [6–8]. In most of the fractional Sturm–Liouville formulations presented recently, the ordinary derivatives in a traditional Sturm–Liouville problem are replaced with fractional derivatives, and the resulting problems are solved using some numerical schemes such as Adomian decomposition method [9], or fractional differential transform method [10], or alternatively using the method of Haar wavelet operational matrix [11]. However, in such numerical studies, round-off errors and the pseudo-spectra introduced in approximating the infinite-dimensional boundary-value problem as a finite-dimensional eigenvalue problem prohibit computing more than a handful of eigenvalues and eigenfunctions with the desired precision. Furthermore, these papers do not examine the common properties of Fractional Sturm–Liouville Problems (FSLPs) such as orthogonality of the eigenfunctions of the fractional operator in addition to the reality or complexity of the eigensolutions.

* Corresponding author.
E-mail address: gk@dam.brown.edu (G.E. Karniadakis).

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.06.031

Establishing the aforementioned fundamental properties for FSLPs is very important in establishing proper numerical methods, e.g. the eigensolutions of the RFSLP may be complex [12]. To this end, some results have been recently provided in [13,14], where the fractional character of the problem has been considered through defining a classical Sturm–Liouville operator, extended by the term that includes a sum of the left- and right-sided fractional derivatives. More recently, a Regular Fractional Sturm–Liouville Problem (RFSLP) of two types has been defined in [15], where it has been shown that the eigenvalues of the problem are real, and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. However, the discreteness and simplicity of the eigenvalues have not been addressed. In addition, the spectral properties of a regular FSLP for diffusion operator have been studied in [16] demonstrating that the fractional diffusion operator is self-adjoint. The recent progress in FSLPs is promising for developing new spectral methods for fractional PDEs, however, the eigensolutions have not been obtained explicitly and no numerical approximation results have been published so far.

The main contribution of this paper is to develop a spectral theory for the Regular and Singular Fractional Sturm–Liouville Problems (RFSLP & SFSLP) and demonstrate its utility by constructing explicitly proper bases for numerical approximations of fractional functions. To this end, we first consider a regular problem of two kinds, i.e., RFSLP-I & -II. Then, we obtain the analytical eigensolutions to these problems explicitly for the first time. We show that the explicit eigenvalues to RFSLP-I & -II are real, discrete and simple. In addition, we demonstrate that the corresponding eigenfunctions are of non-polynomial form, called Jacobi *poly-fractonomials*. We also show that these eigenfunctions are orthogonal and dense in $L^2_w[-1,1]$, forming a complete basis in the Hilbert space. We subsequently extend the regular problem to a singular fractional Sturm–Liouville problem again of two kinds SFSLP-I & -II, and prove that the eigenvalues of these singular problems are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal; these too are computed analytically. We show that the eigensolutions to such singular problems share many fundamental properties with their regular counterparts, with the explicit eigenfunctions of SFSLP-I & -II completing the family of the Jacobi poly-fractonomials. Finally, we complete the spectral theory for the regular and singular FSLPs by analyzing the approximation properties of the eigenfunctions of RFSLPs and SFSLPs, which are employed as basis functions in approximation theory. Our numerical tests verify the theoretical exponential convergence in approximating non-polynomial functions in $L^2_w[-1,1]$. We compare with the standard polynomial basis functions (such as Legendre polynomials) demonstrating the fast exponential convergence of the poly-fractonomial bases.

In the following, we first present some preliminary of fractional calculus in Section 2, and we proceed with the theory on RFSLP and SFSLP in Sections 3 and 4. In Section 5 we present numerical approximations of selected functions and we summarize our results in Section 6.

2. Definitions

Before defining the boundary-value problem, we start with some preliminary definitions of fractional calculus [4]. The left-sided and right-sided Riemann–Liouville integrals of order μ, when $0<\mu<1$, are defined, respectively, as

$$\left({}^{RL}_{x_L}\mathcal{I}^{\mu}_{x} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_{x_L}^{x} \frac{f(s)\,ds}{(x-s)^{1-\mu}}, \quad x > x_L, \tag{1}$$

and

$$\left({}^{RL}_{x}\mathcal{I}^{\mu}_{x_R} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{x_R} \frac{f(s)\,ds}{(s-x)^{1-\mu}}, \quad x < x_R, \tag{2}$$

where Γ represents the Euler gamma function. The corresponding inverse operators, i.e., the left-sided and right-sided fractional derivatives of order μ, are then defined based on (1) and (2), as

$$\left({}^{RL}_{x_L}\mathcal{D}^{\mu}_{x} f\right)(x) = \frac{d}{dx}\left({}^{RL}_{x_L}\mathcal{I}^{1-\mu}_{x} f\right)(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_{x_L}^{x} \frac{f(s)\,ds}{(x-s)^{\mu}}, \quad x > x_L, \tag{3}$$

and

$$\left({}^{RL}_{x}\mathcal{D}^{\mu}_{x_R} f\right)(x) = \frac{-d}{dx}\left({}^{RL}_{x}\mathcal{I}^{1-\mu}_{x_R} f\right)(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{-d}{dx}\right) \int_{x}^{x_R} \frac{f(s)\,ds}{(s-x)^{\mu}}, \quad x < x_R. \tag{4}$$

Furthermore, the corresponding left- and right-sided Caputo derivatives of order $\mu \in (0,1)$ are obtained as

$$\left({}^{C}_{x_L}\mathcal{D}^{\mu}_{x} f\right)(x) = \left({}^{RL}_{x_L}\mathcal{I}^{1-\mu}_{x} \frac{df}{dx}\right)(x) = \frac{1}{\Gamma(1-\mu)} \int_{x_L}^{x} \frac{f'(s)\,ds}{(x-s)^{\mu}}, \quad x > x_L, \tag{5}$$