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High-order entropy stable finite difference schemes for nonlinear conservation laws: Finite domains

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ABSTRACT

Nonlinear entropy stability is used to derive provably stable high-order finite difference operators including boundary closure stencils, for the compressible Navier–Stokes equations. A comparison technique is used to derive a new Entropy Stable Weighted Essentially Non-Oscillatory (SSWENO) finite difference method, appropriate for simulations of problems with shocks. Viscous terms are approximated using conservative, entropy stable, narrow-stencil finite difference operators. The efficacy of the new discrete operators is demonstrated using both smooth and discontinuous test cases.

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1. Introduction

The state of numerical solutions to nonlinear conservation laws is far from complete. While it is commonplace to use high-order numerical methods to calculate efficient and accurate solutions for smooth problems, solutions of problems with shocks are considerably more difficult to simulate. Solution methods for these problems are typically hybrid [1,2] or high-order adaptive [3,4] schemes, or highly dissipative low-order methods. Many methods have been devised that attempt to balance accuracy, added dissipation, and efficiency. Most of these methods are designed using linear analysis of linearized equations that do not admit the formation of shocks and thus do not correctly account for the character of the underlying nonlinear problem. Additionally, stability proofs that rely on linear analysis are dependent on the resolution and do not guarantee stability for under-resolved regions. To overcome these limitations, we seek numerical methods that are based on nonlinear analysis.

An early use of nonlinear analysis appears in the work of Hughes et al. [5] in the context of Galerkin and Petrov–Galerkin finite-element methods (FEM). Entropy stability of the FEM follows immediately if the Navier–Stokes equations are rotated into symmetric form and discretized using the FEM. Honein and Moin [6] use a skew-symmetric nonconservative splitting of the convective terms to achieve enhanced robustness in simulations of smooth compressible turbulence. Olsson and Oliger [7] prove nonlinear stability of the Euler equations using a canonical (yet nonconservative) splitting of the equations and summation-by-parts (SBP) spatial operators. These ideas are extended to include shock capturing by Gerritsen and

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Olsson [8]. Yee et al. [9,10] achieve correct shock locations using a (discretely nonconservative) entropy splitting for the convective fluxes. Although the previously mentioned (high-order) approaches exploit some facet of nonlinear stability analysis to achieve enhanced robustness and accuracy, none of them are discretely consistent with the Lax–Wendroff theorem!

Any numerical method applied to problems that admit shocks should provably recover a weak solution of the conservation law upon convergence [11]; a sufficient condition is the use of a consistent and conservative discretization of the conservation law form of the equations. It should be further proven that the weak solution recovered is the physically realizable entropy solution [11,12]. Convergence to a solution by a consistent, conservative discrete operator satisfying an entropy inequality guarantees an admissible weak solution of the governing equations.[1]

Satisfying an entropy inequality is an uncommon property for a conservative, high-order methods. Recent advances in entropy stability theory for the compressible Euler equations now facilitate the development of conservative and entropy conservative high-order formulations. Entropy conservative schemes are constructed by Tadmor and others [12–15] for second-order finite volume methods. An extension to high-order formulations on periodic domains is given by LeFloch and Rhode [16]. These schemes are made computationally tractable for the Navier–Stokes equations through the work of Ismail and Roe [17]. A methodology for constructing entropy stable schemes satisfying a cell entropy *inequality* and capable of simulating flows with shocks in periodic domains is developed by Fjordholm et al. [18].

Herein, a generalized approach to entropy stability is developed based on SBP operators, that naturally extends to high-order operators on complex domains. First, the general conditions sufficient for achieving entropy conservation are developed based on a generalized summation-by-parts property. Next, the entropy conservative schemes for conservation laws developed by Tadmor and others [13–15] are extended to high-order on finite domains including boundary closures. A comparison technique [13] is then used to develop an entropy stable correction for a high-order WENO algorithm [19]. The combined dissipative algorithm is well suited for simulating problems with shocks. Finally, high-order narrow-stencil [20], discrete operators are derived for the viscous terms and are implemented in a provably entropy stable manner.

Using the methodology developed herein, we demonstrate the robustness and accuracy of the resulting entropy stable WENO operators using Burgers equation and the Euler equations. We also show how schemes developed using linear stability can fail in the presence of a shock by comparing the newly developed entropy stable schemes with the Energy Stable WENO scheme of Fisher et al. [21].

The entropy stable discrete operators developed herein are an important step towards a provably stable simulation methodology of arbitrary order for complex geometries. The extension of the entropy stable numerical methods to generalized (3D) curvilinear coordinates and multi-domain configurations to simulate more complex geometries is included in a companion paper. Two major hurdles remain on the path towards L_2 stability of the compressible Navier–Stokes equations. First, well-posed physical boundary conditions are needed that preserve the entropy stability property of the interior operator. Second, a fully discrete operator is needed that preserves the semi-discrete entropy stability properties (e.g. trapezoidal rule) [13], but that maintains positivity of the density and temperature.

The organization of this document is as follows. The theory of entropy analysis for finite difference methods is detailed in Section 2. Conditions and corresponding methods for satisfying entropy stability on finite domains using arbitrarily high-order accurate finite difference methods are developed in Section 3. The application of these methods to Burgers equation and the compressible Euler and Navier–Stokes equations is illustrated in Section 4. Finally, the accuracy and robustness of the resulting high-order schemes are demonstrated in Section 5 and conclusions are discussed in Section 6.

2. Methodology

In this section, we introduce the theory of entropy stability and define the necessary finite difference nomenclature for conservation laws.

2.1. Nonlinear conservation laws and the entropy condition

The most general form of the one-dimensional inviscid conservation law on a bounded domain is the integral form,

$$\frac{d}{dt}\int_{x_L}^{x_R} q \, \mathrm{d}x + f(q)|_{x_L}^{x_R} = 0, \quad x \in [x_L, x_R], \ t \in [0, \infty), \tag{2.1}$$

where q denotes a scalar or vector of conserved variables, f is the nonlinear flux function, and the domain bounds have been assumed fixed. As noted by Lax [22], solutions satisfying the integral form in (2.1) are generalized or *weak solutions* of the conservation law and do not need to be smooth or even continuous. For smooth problems, the strong differential form of the conservation law can be written as

$$q_t + f(q)_x = 0, \quad x \in [x_L, x_R], \ t \in [0, \infty). \tag{2.2}$$

[1] Note that satisfying only an entropy inequality does not guarantee recovery of an admissible physical solution; e.g., nonconservative equations with an explicit statement of entropy dissipation. Neither does it guarantee that the discrete operator admits a solution; e.g., a conservative and entropy conservative, non-dissipative discrete operator that has no mechanism for providing sufficient dissipation at the shock.

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