Journal of Computational Physics 252 (2013) 591–605

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

An adaptive finite volume method for 2D steady Euler equations with WENO reconstruction

Guanghui Hu

Department of Mathematics, University of Macau, Macao SAR, China

ARTICLE INFO

Article history:
Received 24 February 2013
Received in revised form 3 July 2013
Accepted 7 July 2013
Available online 15 July 2013

Keywords:
Adaptive methods
Finite volume methods
Steady Euler equations
WENO reconstruction
Unstructured grids

ABSTRACT

An adaptive finite volume method for 2D steady Euler equations on unstructured grids is proposed. The framework of the finite volume method for the steady Euler equations follows the one in the paper [G.H. Hu, R. Li, and T. Tang, A robust WENO type finite volume solver for steady Euler equations on unstructured grids, Commun. Comput. Phys. 9 (2011) 627–648]. In this paper, we introduce the mesh adaptive methods to improve the above numerical method. The features of this work include: (i) different reconstruction stencils for WENO reconstruction are discussed in detail, including their performance on the convergence of steady state solutions and on the application of h-adaptive methods, and (ii) an effective indicator for generating quality nonuniform mesh is proposed, which is based on the entropy production. The improvement of the numerical methods is demonstrated by plenty of numerical examples.

1. Introduction

The steady Euler equations are very important in Computational Aerodynamics. For example, when an aircraft is in its cruise state, the distributions of physical variables such as density, velocity, and pressure do not change with the time evolution. In this case, the steady Euler equations serve as the governing equations to depict such cruise state. Since the nonlinearity of the Euler equations and the complexity of the configuration of the problems, it is generally impossible to obtain the analytical solutions for the Euler equations. Consequently, an efficient numerical method becomes crucial for the study of the Euler equations.

The 2D inviscid steady Euler equations are given by

$$\nabla \cdot F(\mathbf{U}) = 0 \tag{1}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \qquad F(\mathbf{U}) = \begin{bmatrix} \rho u & \rho v \\ \rho u^2 + \mathrm{p} & \rho u v \\ \rho u v & \rho v^2 + \mathrm{p} \\ u(E + \mathrm{p}) & v(E + \mathrm{p}) \end{bmatrix}.$$

In the above equations, ρ, $\vec{u} = [u, v]^T$, p, E stand for the density, the velocity, the pressure, and the energy, respectively. $F(\mathbf{U})$ denotes the inviscid flux. The equation of state below is used to close the system,

E-mail address: garyhu@umac.mo.

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.07.006

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\left(u^2 + v^2\right),$$

where $\gamma = 1.4$ stands for the ratio of specific heats of the prefect gas.

There have been many papers on WENO schemes for the Euler equations (1), see, e.g., [24,25,21,8,10,28,9,19]. In [14], a robust finite volume method is proposed for (1) on unstructured grids. In that paper, the steady Euler equations are linearized by the Newton methods. The finite volume methods are employed for the spatial discretization, which numerically guarantees the conservative property of the physical variables. To prevent the nonphysical oscillation and keep the numerical accuracy simultaneously, a WENO reconstruction method for the linear reconstruction is developed. Since the temporal terms in the steady equations are absent, the derived system needs to be regularized. Unlike the traditional methods which add the pseudo-time terms on the steady equations, the authors in [14] use the norm of the local residual in each governing cell for the regularization, which works very well in the implementation. To efficiently solve the final linear system, a geometrical multi-grid method is designed there, and the block lower–upper symmetric Gauss–Seidel iteration is used as the smoother. With this numerical framework, the desired convergence order is observed successfully in the convergence test, and the residual of the system is reduced to the machine accuracy efficiently for the problems with subsonic and transonic configurations. More impressively, the numerical convergence of the method seems like not sensitive to the selection of the parameters in the algorithm, which makes the method more suitable for the simulation with practical configurations.

In this paper, we improve the numerical method proposed in [14] by introducing the mesh adaptive methods. To reach an accurate numerical solution, global refining the mesh is a straightforward way. Specially for the simulation with a shock in the flow field, since the position of the shock is unknown in advance, it seems like that an initial uniform mesh and global refinement is the only method to catch the shock. The mesh adaptive methods give us a second choice, which could also be an economical choice. The adaptive methods mark the trouble region in the computational domain, and implement the mesh refinement just in the trouble region. With this method, computational resource is effectively saved, while a quality numerical solution can be expected at the same time.

There are mainly two kinds of mesh adaptive methods. One is *r*-adaptive methods which relocates the mesh grids without changing the number of mesh grids. The other one is *h*-adaptive methods which locally refine and/or coarsen the mesh grids. The *r*-adaptive methods have been widely used in, for example, computational fluid dynamics [23]. We refer to [15] and references therein for more details and applications of the *r*-adaptive methods. It is worth mentioning that for the *r*-adaptive methods which are based on the harmonic maps, it is nontrivial to adopt such *r*-adaptive methods for the numerical experiments in this paper. As we know, to guarantee the existence and uniqueness of the harmonic map, the domain we used in the simulation needs to be a simply connected region, which is not the case in this paper. To avoid this problem, the domain decomposition methods could be one reasonable choice.

Since their success in electronic structure calculations [3,26], computational electromagnetics [4,2], etc., we use the *h*-adaptive methods in this paper to improve the numerical method in [14]. To realize the *h*-adaptive methods in an efficient way, a quality data structure and a well-designed error indicator are necessary. The data structure is used to flexibly handle the mesh local refinement and/or coarsening, and the error indicator marks the trouble region in the computational domain. In this paper, we will follow [16] to use the hierarchical geometry tree (HGT) to manage the mesh structure, and to implement the mesh refinement and/or coarsening. Regarding the error indicator, a possible method is the *a posteriori* error estimation. Although there are a lot of works [29,20,7] concerning the *a posteriori* error estimation methods on the finite element methods, the frameworks in those papers cannot be trivially used for the finite volume methods. In [5,11], the goal-oriented *a posteriori* error estimation methods are proposed. However, to make the error indicator a reliable one, a dual problem needs to be solved with higher-order numerical methods, which may cause the problem on the computational cost. In this paper, an *ad hoc* indicator for Euler equations, which is from the entropy production, is presented and detailedly discussed. The reliability of this indicator is checked by a variety of numerical tests.

In [14], a WENO reconstruction method, which is different from the traditional ones [12,22], is adopted for the solution reconstruction. The difference is that a new reconstruction patch is proposed in [14]. In this paper, a detailed discussion on the behaviors of these patches for the steady problem is presented, which shows that the patch in [14] is a better choice when the mesh adaptive method is adopted. Besides the reconstruction patch, a parameter appeared in the smoothness indicator of the polynomials is also tested carefully, and an appropriate range of this parameter for the steady problems is given. From the comparison of the numerical results from different patches, the one proposed in [14] is relatively not sensitive to the parameters in the WENO methods. The numerical results also show that with this patch, the convergence of the steady state of the Euler equations can still be reached successfully by using the adaptive method, which is not the case for other reconstruction patches.

The rest of the paper is organized as follows. In next section, the WENO reconstruction for the steady problems is detailedly discussed. Then the *h*-adaptive methods we used in this paper is introduced in Section 3. In Section 4, the reliability and the effectiveness of our adaptive method is checked by a variety of numerical experiments. Finally, the concluding remarks of this paper is given in Section 5.

2. WENO reconstructions on unstructured grids

A quite standard way to implement the WENO reconstruction is as follows. First, one needs to choose a reconstruction patch for an element, and determine a set of reconstruction stencils on this patch. The sizes of the patch and each stencil