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A kernel-free boundary integral method for implicitly defined surfaces

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ABSTRACT

The kernel-free boundary integral (KFBI) method is a structured grid method for general elliptic partial differential equations. Unlike the standard boundary integral method, it avoids direct evaluation of volume and boundary integrals, which needs to know analytical expressions for the integral kernels. To evaluate a boundary or volume integral, the KFBI method first solves a corrected interface problem on a structured grid and then the numerical solution on the structured grid is interpolated to get approximate values of the integral at points on the boundary. Selection of control points of the boundary plays a key role in the KFBI method since both the correction for the interface equations and the interpolation with the structured grid based solution involve calculation of tangential derivatives of boundary data while stability and efficiency of the numerical differentiation critically depend on the distribution of control points. This work proposes a new point selection method, based on an overlapping surface decomposition of the boundary, which is implicitly defined by a level set function. The points selected are intersection points of the boundary with the grid lines of an underlying Cartesian grid. By the method, the interpolation stencils can be easily chosen to be locally uniform along a coordinate axis in two space dimensions and locally uniform on a coordinate plane in three space dimensions, which allows efficient numerical differentiation and boundary reconstruction/representation. An additional equilibrating process of boundary data further guarantees stable numerical differentiation. Numerical results demonstrating the method with examples in both two and three space dimensions are presented.

1. Introduction

Structured grid methods gain their popularity for elliptic partial differential equations on complex domains, especially for free boundary and moving interface problems, because of the convenience in grid generation and the availability of robust and fast solvers for the resulting discrete equations. Representative structured grid methods are the phase field method by J.W. Cahn and J.E. Hilliard [7], the capacitance matrix method by W. Proskurowski and O. Widlund [29], the immersed boundary (IB) method by C.S. Peskin [26–28], the immersed interface (II) method by R.J. LeVeque and Z. Li [13], Z. Li and K. Ito [15], the ghost of fluid (GF) method by S. Osher et al. [16,17,25,30], the volume of fluid (VOF) method by Johansen and Colella et al. [10,23]. The grid-based boundary integral methods by A. Mayo et al. [19–22,24] and J.T. Beale et al. [2–4] can also be classified as structured grid methods.

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The kernel-free boundary integral (KFBI) method is also a structured grid method for general elliptic partial differential equations. It is a direct extension of Mayo's grid-based boundary integral method [2,3,19] and in spirit similar to Li's augmented strategy for constant coefficient problems [14,15], Wiegmann and Bube's explicit jump II method [32] and Calhoun's Cartesian grid method [8]. The most obvious difference of the KFBI method from others is that it works with more general elliptic operators with possible anisotropy and inhomogeneity.

Unlike other structured grid methods such as the ghost of fluid method, the decomposed II method by Berthelsen [5], the matched interface and boundary (MIB) method by Zhou et al. [36], the KFBI method preserves the symmetry and positive definiteness of the coefficient matrix of the discrete system. The resulting linear system can be solved efficiently by a fast and reliable solver such as the fast Fourier transform for constant coefficient problems or the geometric multigrid method for variable coefficients problems.

The KFBI method avoids direct evaluation of a volume or boundary integral, which traditionally needs to know an analytical expression for the integral kernel. Here, we remark that the kernel-independent fast multipole method (KIFMM), proposed by Ying, Biros and Zorin [34], is based on kernel evaluations. It does not need an analytical expression for the integral kernel in the far field computation but it does need one in the near field computation. To evaluate a boundary or volume integral, the KFBI method first solves a corrected interface problem on a structured grid and then the numerical solution on the structured grid is interpolated to get approximate values of the integral at points on the boundary. The KFBI method at all does not need to know any analytical expression for the integral kernel. For this, the method is said to be kernel-free. It even works for more general variable coefficients elliptic operators, to which the kernel of the corresponding integral operator is in general unknown or at least difficult to get an analytical expression.

Selection of control points of the boundary plays a key role in the KFBI method since both the correction for the interface equations and the interpolation with the structured grid based solution involve calculation of tangential derivatives of boundary data while stability and efficiency of the numerical differentiation critically depend on the distribution of control points.

This work proposes a new point selection method, based on an overlapping surface decomposition of the boundary, which was originally proposed by Wilson for computing integrals on implicitly defined curves and surfaces [33]. The points selected are intersection points of the boundary with the grid lines of an underlying Cartesian grid. By the method, the interpolation stencils can be easily chosen to be locally uniform along a coordinate axis in two space dimensions and locally uniform on a coordinate plane in three space dimensions, which allows efficient numerical differentiation and boundary reconstruction/representation. An additional equilibrating process of boundary data further guarantees stable numerical differentiation. The point selection method is specifically for implicit surfaces defined by level set functions, and will be good for problems with moving boundaries, especially when topological changes occur.

The remainder of the paper is organized as follows. In Section 2, the kernel-free boundary integral method is briefly introduced and summarized. Section 3 gives the details of the selection method for control points of the boundary, based on the overlapping surface decomposition. Interpolation on the overlapping control points is described in Section 4. In Section 5, the process of equilibrating boundary data for two iterative methods is described. Numerical results demonstrating the method with examples in both two and three space dimensions are presented in Section 6. Finally, the advantages and possible improvement for the KFBI method proposed in this work are discussed in Section 7.

2. Kernel-free boundary integral method

The kernel-free boundary integral method was proposed for general elliptic partial differential equations, including those with variable or/and anisotropic coefficients, nonhomogeneous source terms and different boundary conditions [35]. For simplicity of description, however, in this paper the KFBI method is only summarized and introduced for the Laplace equation subject to the Dirichlet boundary condition.

Let $d = 2$ or 3 be the space dimension and $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. Consider the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \tag{1}$$

subject to the Dirichlet boundary condition

$$u = g \quad \text{on } \Gamma \tag{2}$$

with $g = g(\mathbf{p})$ be a given smooth data. Here, $\mathbf{p} \in \mathbb{R}^d$ is a space point.

Let $\mathcal{B}$ be a larger regular box, which embeds the domain Ω such that their boundaries are disjoint, i.e., $\partial\Omega \cap \partial\mathcal{B} = \emptyset$. The box can be a rectangle or of other shape as long as it can be easily partitioned into a hierarchy of structured grids. Let $G(\mathbf{q}, \mathbf{p})$ be Green's function of the Laplacian operator on the box, which satisfies

$$\Delta G(\mathbf{q}, \mathbf{p}) = \delta(\mathbf{q} - \mathbf{p}) \quad \text{in } \mathcal{B},$$

$$G(\mathbf{q}, \mathbf{p}) = 0 \quad \text{on } \partial\mathcal{B},$$