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# A small stencil and extremum-preserving scheme for anisotropic diffusion problems on arbitrary 2D and 3D meshes ☆

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ABSTRACT

In this paper a nonlinear extremum-preserving scheme for the heterogeneous and anisotropic diffusion problems is proposed on general 2D and 3D meshes through a certain linearity-preserving approach. The so-called harmonic averaging points located at the interface of heterogeneity are employed to define the auxiliary unknowns. This new scheme is locally conservative, has only cell-centered unknowns and possesses a small stencil, which is five-point on the structured quadrilateral meshes and seven-point on the structured hexahedral meshes. The stability result in $H_1$ norm is obtained under quite general assumptions. Numerical results show that our scheme is robust and extremum-preserving, and the optimal convergence rates are verified on general distorted meshes in case that the diffusion tensor is taken to be anisotropic, at times discontinuous.



## 1. Introduction

Anisotropic diffusion problem arises in a wide range of scientific fields such as hydrogeology, oil reservoir simulations, plasma physics, semiconductor modeling, and so on. In many cases, diffusion equation is coupled with some other physical processes such as the Lagrange approach in radiation hydrodynamics. In this case, the solution should respect the physical bounds and the computational mesh may be nonconforming and highly distorted, which makes the design of numerical schemes even difficult. Accurate modeling of diffusion processes in these applications requires reliable discretization methods.

In this article, we are interested in constructing a cell-centered finite volume scheme which satisfies the following:

- it is locally conservative;
- it satisfies discrete extremum principle;
- it has a local and small stencil;
- it is simple and easy for coding especially in three dimensions;
- it must be reliable on unstructured anisotropic meshes that may be highly distorted;
- it allows heterogeneous full diffusion tensors;
- it has a second-order accuracy for smooth solutions.

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The second property is the most difficult one for a discrete scheme to satisfy, which includes the maximum principle and minimum principle. A close related concept is the so-called monotone scheme that preserves the nonnegativity of the solution, a special case of the minimum principle. Extremum principle or monotonicity is very important, for example, for diffusion terms in modeling two-phase flows in porous media [22] and for coupling transport equation with a chemical model.

It is well known that classical linear schemes do not always satisfy extremum principle or monotonicity for distorted meshes or with high anisotropy ratio [15]. To our knowledge, there are no linear schemes unconditionally satisfying all of the above requirements. There are several linear schemes [1,7,13,25,28] satisfying some requirements above, but not all of them. The schemes which satisfy the discrete extremum principle impose severe restrictions on the meshes or the diffusion coefficients. To enlarge the class of admissible problems and meshes, some schemes such as the multi-point flux approximation methods (MPFA) [2] use built-in flexibility to increase their monotonicity regions. For classical finite elements, it is explained in [9], that for the Laplacian, the resulting global matrix is an M-matrix if some geometrical constraints are satisfied. Monotone schemes based on slope limiters are proposed in [25]. Two approaches based on repair technique and constrained optimization have been introduced to enforce discrete extremum principle for linear finite element solutions on triangular meshes [20]. A linear scheme [24] satisfying a maximum principle for anisotropic diffusion operators on distorted grids is developed, but this method is generally only first order accurate for smooth solutions. The sufficient condition to ensure the monotonicity of the mimetic finite difference method is analyzed in [17].

Recently, a few nonlinear schemes [6,8,23,19] have been proposed. A nonlinear finite volume scheme proposed in [23] satisfying either the minimum or the maximum principle but not the both simultaneously. The monotonicity of this method for steady state diffusion problems was proven in [18], and 3D extension for this method has been proposed and analyzed in [16]. Further development of the method was made in [26,27,33]. A common property of these methods is that in addition to primary unknowns defined at cell centers, solution values at mesh vertices or edge midpoints are involved. These auxiliary unknowns are usually interpolated from primary cell-centered unknowns. The interpolation technique proves to be a very important issue in these methods and it becomes a very difficult one when the diffusion tensor is discontinuous or polyhedral meshes are involved. The authors in [33] used vertex unknowns as auxiliary unknowns and suggested a piecewise linear approximation to the solution around points where the coefficient is discontinuous. However, as shown in [18,33], the choice of the interpolation method affects the accuracy of the nonlinear scheme even in the case of a constant diffusion coefficient. In [26], a nonlinear extremum-preserving finite volume scheme is proposed, and the edge midpoints are used to define auxiliary unknowns. The interpolation procedure requires that each edge midpoint is located within a triangle formed by three cell centers and located in the smooth area of the solution. When the edge midpoint is on the discontinuity, some special technique is required. Therefore, as pointed out in Remark 1 of [26], this interpolation procedure can be used to deal with only a part of discontinuous diffusion problems. In [27], the cell edge unknowns were used once again as auxiliary ones in the construction of a monotone finite volume scheme. This time, following the idea of MPFA [1], the cell edge unknowns are eliminated by solving a local linear system. However, the solvability of such local linear systems cannot be always guaranteed theoretically [28]. The most important thing is that we do not know whether this interpolation method is positivity-preserving, which is one of the fundamental requirements in the construction of the extremum-preserving or monotone schemes. In most cases, the construction of a second-order positivity-preserving interpolation algorithm is a more challenging task than the design of interpolation-based monotone schemes itself. This is the reason for the authors in [19] to suggest a certain interpolation-free monotone scheme, however, this alternative approach introduces a constraint on the choice of cell centers.

In this article, we further develop and analyze the nonlinear finite volume scheme proposed in [26,27,33]. One key characteristics of our new scheme is that a new and simple interpolation technique is employed to improve the robustness of the scheme for strong heterogeneous and anisotropic diffusion problems and make our scheme to have a small stencil. We introduce, for each cell facet, the so-called harmonic averaging point suggested in [4,13]. Unknowns located at these particular cell facet points are utilized as auxiliary unknowns, and can be interpolated from the two cell-centered unknowns which share this cell facet. The use of harmonic averaging points not only simplifies the interpolation procedure, but also assures it to be a positivity-preserving one. Usually each cell facet has a harmonic averaging point, which allows our one-side flux expression to process a small stencil involving only the present cell and the cells having a common facet with it. This nature makes it easy to implement our scheme on arbitrary and unstructured polygonal meshes or to extend the scheme to three-dimensional polyhedral meshes. The implementation for both 2D and 3D case is almost the same, most of the program modules are in common use, and only some particular modules such as the computation of cell matrix, cell volume and facet area should be written separately. Moreover, the present approach allows us to obtain the stability result in $H_1$ norm for the new scheme.

Our new scheme satisfies all the seven properties mentioned above, and it should be noted that: (1) our scheme has a small stencil for the interior cell-centered unknowns, i.e., a *four-point* stencil on the triangle meshes, a *five-point* stencil on the structured quadrilateral meshes, and a *seven-point* stencil on structured hexahedral meshes; (2) it is *linearity-preserving*, which means that the scheme provides the exact solution whenever, on each mesh cell, the solution is linear and the diffusion coefficient is constant. This property can be found in many articles for instance in [1,4,13,14,28–31].

The outline of the paper is organized as follows. In Section 2, we state the diffusion problem and give some notations. In Section 3, A small stencil and extremum-preserving finite volume scheme is constructed in five steps. The stability analysis in $H_1$ norm is given in Section 4. Then in Section 5, we present some 2D and 3D numerical experiments to illustrate the