Journal of Computational Physics 250 (2013) 332–346

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A homotopy method based on WENO schemes for solving steady state problems of hyperbolic conservation laws

Wenrui Hao [a], Jonathan D. Hauenstein [b], Chi-Wang Shu [c], Andrew J. Sommese [a], Zhiliang Xu [a], Yong-Tao Zhang [a,*]

[a] Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556, United States
[b] Department of Mathematics, North Carolina State University, Raleigh, NC 27695, United States
[c] Division of Applied Mathematics, Brown University, Providence, RI 02912, United States

ARTICLE INFO

Article history:
Received 4 September 2012
Received in revised form 19 January 2013
Accepted 8 May 2013
Available online 27 May 2013

Keywords:
Homotopy continuation
Hyperbolic conservation laws
WENO scheme
Steady state problems

ABSTRACT

Homotopy continuation is an efficient tool for solving polynomial systems. Its efficiency relies on utilizing adaptive stepsize and adaptive precision path tracking, and endgames. In this article, we apply homotopy continuation to solve steady state problems of hyperbolic conservation laws. A third-order accurate finite difference weighted essentially non-oscillatory (WENO) scheme with Lax–Friedrichs flux splitting is utilized to derive the difference equation. This new approach is free of the CFL condition constraint. Extensive numerical examples in both scalar and system test problems in one and two dimensions demonstrate the efficiency and robustness of the new method.

1. Introduction

Numerical simulation of hyperbolic conservation laws has been a major research and application area of computational mathematics in the last few decades. Weighted essentially non-oscillatory (WENO) finite difference/volume schemes are a popular class of high order numerical methods for solving hyperbolic partial differential equations (PDEs). They have the advantage of attaining uniform high order accuracy in smooth regions of the solution while maintaining sharp and essentially monotone transitions of discontinuities. The first WENO scheme was constructed in [21] for a third-order accurate finite volume version. In [19], third- and fifth-order accurate finite difference WENO schemes in multi-space dimensions were constructed, with a general framework for the design of the smoothness indicators and nonlinear weights. Later, WENO schemes on unstructured meshes were developed for dealing with complex domain geometries [14,36]. For steady state problems of hyperbolic conservation laws, efficiently solving the large nonlinear system derived from the WENO discretization is still a challenging problem.

A popular approach for solving steady state problems is the time marching method. Starting from an initial condition, the numerical solution evolves into a steady state by using a time stepping scheme. For examples using this kind of approach, see the numerical experiments in [19] and high order residual distribution (RD) conservative finite difference WENO schemes for solving the steady state problems in [7]. The class of RD schemes for solving steady state problems were developed in, e.g.,

* Corresponding author. Tel.: +1 574 631 6079; fax: +1 574 631 4822.
E-mail addresses: whao@nd.edu (W. Hao), hauenstein@ncsu.edu (J.D. Hauenstein), shu@dam.brown.edu (C.-W. Shu), sommese@nd.edu (A.J. Sommese), zxu2@nd.edu (Z. Xu), yzhang10@nd.edu (Y.-T. Zhang).
URLs: http://www.nd.edu/~whao (W. Hao), http://www.math.ncsu.edu/~jdhauens (J.D. Hauenstein), http://www.dam.brown.edu/people/shu/ (C.-W. Shu), http://www.nd.edu/~sommese (A.J. Sommese), http://www.nd.edu/~yzhang10 (Y.-T. Zhang).

0021-9991/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jcp.2013.05.008

[8,25,26,2,1]. They use a similar pointwise representation of the solution as in finite difference schemes, but allow conservative approximations with high order accuracy on very general meshes. In [7], a high order accurate residual by a WENO integration procedure was computed directly and the resulting RD-WENO schemes achieved both high order accuracy and conservativeness on arbitrary Cartesian or curvilinear meshes without any smoothness assumption.

A big advantage of the time marching method is that the computed steady state is stable and usually carries physical properties of the system and the initial condition. However, from the point of view of computational efficiency, the computational cost of time marching method for obtaining a steady state solution is *not* linear due to restricted time-step sizes by the well-known Courant–Friedrichs–Lewy (CFL) condition [10]. One way to improve this is to take advantage of the properties of hyperbolic PDEs, i.e., information propagates along characteristics from boundaries of a steady state problem. In [37,34] a class of iterative methods, called "fast sweeping methods" [38], have been combined with WENO schemes to accelerate the convergence of the time marching approach. Fast sweeping methods utilize alternating sweeping strategy to cover a family of characteristics in a certain direction simultaneously in each sweeping order. Coupled with the Gauss–Seidel iterations, the methods can achieve a fast convergence speed for computations of steady state solutions. Recently, this acceleration approach has been applied for the case of hyperbolic conservation laws in [6]. Another way to improve the convergence is to design new smoothness indicators. In [35], a new smoothness indicator is introduced to remove the slight post-shock oscillations and the numerical residue so that the WENO schemes can converge to machine precision.

It is desirable to design a numerical method for solving steady state problems with linear computational complexity. Such methods need to be free of the CFL conditions. For the large nonlinear system derived from the WENO discretization, one way to solve this system is to apply Newton iterations. A Newton iteration based method was adopted to solve the steady two dimensional Euler equations in [15,16,18]. The matrix-free Squared Preconditioning is applied to Newton iterations nonlinearly preconditioned by means of the flow solver in [17]. In this paper, we design a novel method to solve the large nonlinear system directly. This new method has linear computational complexity and is free of CFL conditions, as shown in our numerical experiments.

Discretizing many systems of nonlinear differential equations produces sparse polynomial systems. Numerical algorithms based on techniques arising in algebraic geometry, collectively called *numerical algebraic geometry*, have been developed to solve polynomial systems. Over the last decade, numerical algebraic geometry (see [20,29,32] for some background), which grew out of continuation methods for finding all isolated solutions of systems of nonlinear multivariate polynomials, has reached a high level of sophistication. Even though the polynomial systems that arise by discretizing differential equation system are many orders of magnitude larger than the polynomial systems that the algorithms of numerical algebraic geometry have been applied to, these algorithms can still be used efficiently to investigate such polynomial systems.

The major tool in numerical algebraic geometry is homotopy continuation. For a given system of polynomial equations to be solved, a homotopy between the given system and a new system (which is easier to solve and share many features with the former system) can be constructed (see Section 3 for a detailed description of this method in this context). Then, one tracks paths starting from each solution of the new system as one moves towards the original system along the homotopy, thereby obtaining solutions of the original system. The homotopy method computes all the complex (which obviously include real) solutions of a system which is known to have only isolated solutions. In this paper, we utilize a homotopy continuation approach to compute steady states of hyperbolic systems and demonstrate that this new approach can easily handle singular systems and also be used to find multiple steady states. The numerical experiments show that the homotopy method is competitive with the Newton based methods [15,16,18] and is faster than the classical time marching methods.

The organization of the article is as follows. We propose a homotopy method based on a third order finite difference WENO scheme in Section 2. In Section 3, we describe homotopy continuation and endgames. Extensive numerical simulation results are contained in Section 4 for one- and two-dimensional scalar and system steady state problems to demonstrate the behavior of our scheme. We conclude in Section 5.

2. Numerical methods

In this article, we solve both one-dimensional and two-dimensional steady state hyperbolic conservation laws. We use a third order accurate finite difference WENO scheme with Lax–Friedrichs flux splitting to discretize the PDEs. The advantage of using a finite difference WENO scheme is that we can perform the WENO reconstructions in a dimension-by-dimension manner, to achieve better efficiency than a finite volume WENO scheme in multi-dimensions [19]. For simplicity, we describe the scheme for solving one-dimensional problems. To solve multi-dimensional problems, one can simply use a dimension-by-dimension approach.

Consider the following one-dimensional hyperbolic conservation laws

$$u_t + (f(u))_x = g(u,x).$$

Setting u_t to zero, the steady state problem becomes

$$(f(u))_x - g(u,x) = 0.$$

For an initial condition u^0, we introduce the homotopy

$$H(u,\epsilon) = \big((f(u))_x - g(u,x) - \epsilon u_{xx}\big)(1-\epsilon) + \epsilon(u-u^0) \equiv 0, \tag{2.1}$$