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Accuracy of the weighted essentially non-oscillatory conservative finite difference schemes

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ABSTRACT

In the reconstruction step of $(2r-1)$ order weighted essentially non-oscillatory conservative finite difference schemes (WENO) for solving hyperbolic conservation laws, nonlinear weights α_k and ω_k, such as the WENO-JS weights by Jiang et al. and the WENO-Z weights by Borges et al., are designed to recover the formal $(2r-1)$ order (optimal order) of the upwinded central finite difference scheme when the solution is sufficiently smooth. The smoothness of the solution is determined by the lower order local smoothness indicators β_k in each substencil. These nonlinear weight formulations share two important free parameters in common: the power p, which controls the amount of numerical dissipation, and the sensitivity ε, which is added to β_k to avoid a division by zero in the denominator of α_k. However, ε also plays a role affecting the order of accuracy of WENO schemes, especially in the presence of critical points. It was recently shown that, for any design order $(2r-1)$, ε should be of $\Omega(\Delta x^2)$ ($\Omega(\Delta x^m)$ means that $\varepsilon \geqslant C\Delta x^m$ for some C independent of Δx, as $\Delta x \to 0$) for the WENO-JS scheme to achieve the optimal order, regardless of critical points. In this paper, we derive an alternative proof of the sufficient condition using special properties of β_k. Moreover, it is unknown if the WENO-Z scheme should obey the same condition on ε. Here, using same special properties of β_k, we prove that in fact the optimal order of the WENO-Z scheme can be guaranteed with a much weaker condition $\varepsilon = \Omega(\Delta x^m)$, where $m(r,p) \geqslant 2$ is the optimal sensitivity order, regardless of critical points. Both theoretical results are confirmed numerically on smooth functions with arbitrary order of critical points. This is a highly desirable feature, as illustrated with the Lax problem and the Mach 3 shock-density wave interaction of one dimensional Euler equations, for a smaller ε allows a better essentially non-oscillatory shock capturing as it does not over-dominate over the size of β_k. We also show that numerical oscillations can be further attenuated by increasing the power parameter $2 \leqslant p \leqslant r-1$, at the cost of increased numerical dissipation. Compact formulas of β_k for WENO schemes are also presented.

1. Introduction

Weighted Essentially Non-Oscillatory (WENO) conservative finite difference schemes [3,4,11,20] are a popular choice for solving nonlinear hyperbolic partial differential equations, especially problems whose solutions contain both discontinuities and small scale smooth structures. The adaptive nonlinear reconstruction procedure of WENO schemes avoids interpolations across nonsmooth data, essentially preventing spurious oscillations near discontinuities in the numerical solution.

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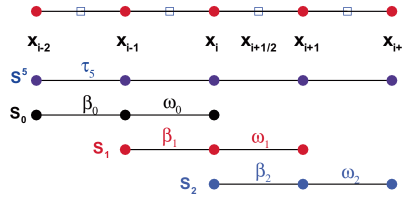

Fig. 1. The computational uniformly spaced grid, with cell centers x_i and cell boundaries $x_{i+\frac{1}{2}}$, and the 5-points stencil S^5, composed of three 3-points substencils S_0, S_1, S_2, used in the fifth-order WENO reconstruction step.

WENO schemes were first introduced in [17] as an improvement to the Essentially Non-Oscillatory (ENO) schemes [9,10,21]. Consider a global stencil S^{2r-1} containing $(2r-1)$ uniformly spaced grid points and its r possible r points substencils $S_k,\ k = 0, \ldots, r-1$ with $S^{2r-1} = \bigcup_{k=0}^{r-1} S_k$. Here the parameter $k = 0, \ldots, r-1$ is known as the shift parameter. Fig. 1 gives an example of the fifth order ($r = 3$) global stencil S^5 and substencils S_1, S_2 and S_3. The ENO reconstruction step uses a nonlinear adaptive conditional procedure to search for a substencil S_k which would give the *smoothest* $(r-1)$ degree local polynomial approximation $P_{r-1}^k(x)$ of the underlying function $f(x)$. The WENO reconstruction procedure, by contrast, assigns a nonlinear weight ω_k to $P_{r-1}^k(x)$ in each substencil S_k, according to a normalized Sobolev norm of $P_{r-1}^k(x)$, which measures its relative smoothness, effectively using the whole $(2r-1)$ points global stencil S^{2r-1}. On the function approximation level, the nonlinear weights ω_k are designed to yield an essentially zero value in S_k where $f(x)$ is discontinuous, emulating the ENO conditional procedure; and to recover the $(2r-1)$ formal order of accuracy (optimal order) in the $(2r-1)$ points global stencil S^{2r-1} when $f(x)$ is sufficiently smooth.

In [14], the authors proposed the nonlinear weights

$$\alpha_k = \frac{d_k}{(\beta_k + \varepsilon)^p}, \quad \omega_k = \frac{\alpha_k}{\sum_{j=0}^{r-1} \alpha_j}, \tag{1}$$

where ε and p are the sensitivity and power parameters, respectively, β_k are the smoothness indicators which measure the regularity of $f(x)$ in L^2 norm in each substencil, and d_k are the ideal weights (see Section 3 for details). This scheme is known as the WENO-JS scheme. The parameter ε was used for avoiding a division by zero in the denominator, and it was expected to have a small constant value so as not to interfere with the smoothness detection of β_k.

However, in [11], it was noticed that the order of accuracy of the WENO-JS scheme was greatly reduced in the presence of critical points if ε was chosen to be too small ($\varepsilon = 10^{-40}$). To overcome this deficiency, the authors proposed a mapping function which, when applied to the nonlinear weights (1), made the WENO-JS scheme to converge with optimal order even with small ε, at the cost of $\sim 25\%$ increase in CPU time. This scheme is known as the WENO-M scheme.

In [3,4],[1] the authors proposed a new formula for the weights,

$$\alpha_k^Z = d_k\left(1 + \left(\frac{\tau_{2r-1}}{\beta_k + \varepsilon}\right)^p\right), \quad \omega_k^Z = \frac{\alpha_k^Z}{\sum_{j=0}^{r-1} \alpha_j^Z}, \tag{2}$$

with an introduction of the global optimal order smoothness indicator τ_{2r-1}, that is a linear combination of the lower order local smoothness indicators β_k which is of higher order and uses the global stencil of $(2r-1)$ points (see Section 3 for details). This scheme is known in the literature as the WENO-Z scheme. The nonlinear weights (2) allowed the WENO-Z scheme to achieve the optimal order without resorting to a mapping function, even with small ε. Analysis on the weights and numerical experiments demonstrated that WENO-Z is less dissipative and has increased resolution power in comparison to the WENO-JS scheme of same order $(2r-1)$ and parameters p and ε.

Nevertheless, in the presence of critical points of high order, where the derivatives of the function $f(x)$ up to some higher order at a given point are all zero (see Definition 1), both the WENO-M and WENO-Z schemes will fail to achieve the optimal order if a small constant ε is used [3,4,11]. The use of larger values for the sensitivity parameter ($\varepsilon = 10^{-6}$) is not advised, though, as it makes both schemes less sensitive to discontinuities and, hence, more prone to generating spurious oscillations. It should be noted here that the analysis on the formal order of accuracy of the WENO reconstruction procedure was performed, with an exception of [1] for the WENO-JS scheme, usually under the assumption of $\varepsilon = 0$.

As a solution for these issues, several authors [1,4,23] have adopted the use of ε as a function of Δx, instead of a fixed constant value. The choice of $\varepsilon = \Omega(\Delta x^2)$ (see Definition 2) was shown to guarantee the convergence with the optimal order $(2r-1)$ for both the WENO-JS and WENO-Z schemes, even at critical points, while keeping the ENO property near discontinuities. It also eliminated the need of using a costly mapping function of the WENO-M scheme to correct the nonlinear weights of the WENO-JS scheme (1).

[1] The 12th most cited JCP article published since 2008 according to SciVerse Scopus.