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Numerical treatment for solving the perturbed fractional PDEs using hybrid techniques

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ABSTRACT

In this article, an accurate numerical approach is introduced. In this approach we mixed between the fractional finite difference method and the restrictive Taylor approximation (RTA). The proposed method is implemented to solve numerically the perturbed fractional partial differential equations (FPDEs). Special attention is given to study the stability and consistency of the method by means of Gerschgorin theorem and using the stability matrix analysis. Two numerical examples are given and compared with the exact solution.

1. Introduction

FPDEs have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, visco-elasticity, biology, physics and engineering [1]. Consequently, considerable attention has been given to the solutions of FPDEs of physical interest. And is known that it is difficult to obtain the exact solution of FPDEs, so approximate and numerical techniques [2–6,8–18] must be used. There are several methods for obtaining the analytical and numerical solutions of PDEs, such as, spectral collocation methods [22,24], finite difference method [21,23,25] and others.

In this section, some basic definitions for fractional derivatives and Gerschgorin theorem which are used in the other sections are given.

Definition 1. The Riemann–Liouville fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(x) := \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x \frac{f(t)}{(x-t)^{\alpha-m+1}}dt, \quad \alpha > 0,\ x > 0,$$

where $m-1 < \alpha \leqslant m$, $m \in \mathbb{N}$, and Γ is Gamma function. For the Riemann–Liouville's derivative we have [19]

$$D^\alpha x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil\alpha\rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geqslant \lceil\alpha\rceil. \end{cases} \tag{1}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Recall that for $\alpha \in \mathbb{N}$, the Riemann–Liouville differential operator coincides with the usual differential operator of integer order.

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Similar to integer-order differentiation, Riemann–Liouville fractional derivative operator is a linear operation

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x),$$

where λ and μ are constants.
For more details on fractional derivatives definitions and its properties see [19,20].

Theorem 1 (*Gerschgorin theorem*). *Let the matrix $A \equiv (a_{ij})$ has eigenvalues λ and define the absolute row and column sums by*

$$r_i \equiv \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad c_i \equiv \sum_{i=1, i\neq j}^{n} |a_{ij}|,$$

then

(a) *Each eigenvalues lies in the union of the row circles R_i, $i = 1, 2, \ldots, n$ where*
$R_i \equiv \{z : |z - a_{ii}| \leqslant r_i\}$;

(b) *Each eigenvalues lies in the union of the column circles C_j, $j = 1, 2, \ldots, n$ where*
$C_j \equiv \{z : |z - a_{jj}| \leqslant c_j\}$.

Proof. See [7]. □

In this paper, we will introduce the numerical solution of the following perturbed fractional partial differential equation of the form

$$\delta \frac{\partial u}{\partial t} - k D^{\alpha} u = s(x), \quad a < x < b, \ t > 0, \ 1 < \alpha \leqslant 2, \tag{2}$$

where $u = u(x,t)$, $\delta \ll 1$ (very small) and k are given positive constants and the parameter α refers to the order of Riemann–Liouville derivative of spatial. The function $s(x)$ is a given heat source. We assume an initial condition

$$u(x,0) = u^0(x), \quad \text{for } a \leqslant x \leqslant b,$$

and zero Dirichlet boundary conditions.
Note that, at $\alpha = 2$, Eq. (2) is the classical perturbed parabolic PDE

$$\delta \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = s(x).$$

2. Preliminaries and basic definitions

2.1. Restrictive Taylor approximation

In this section, we will use the restrictive Taylor's expansion [4,5] of the function $f(x)$ at the point x_0, with parameter θ to be determined such that the error equal to zero at certain point a. If the parameter θ reduces to one we get the classical Taylor's expansion at the point x_0. The advantage is that it has exact value at the points a and x_0 and relatively small in between.

Consider the function $f(x)$ defined in a neighborhood of x_0, and has derivative up to order $n+1$, we define the restrictive Taylor's approximation $RT_{n,f(x)}(x)$ of degree n for the function $f(x)$ at the point x_0 as follows [5]

$$RT_{n,f(x)}(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x-x_0)^{n-1}}{(n-1)!} f^{(n-1)}(x_0) + \frac{\theta (x-x_0)^n}{n!} f^{(n)}(x_0). \tag{3}$$

The restrictive parameter θ is to be determined, such that

$$RT_{n,f(x)}(a) = f(a). \tag{4}$$

It means that the considered approximation is exact at two points x_0 and a. Let us suppose

$$f(x) = RT_{n,f(x)}(x) + \boldsymbol{R}_{n+1}(x, \theta(x)), \tag{5}$$

where $\boldsymbol{R}_{n+1}(x,\theta(x))$ is the remainder term of the restrictive Taylor's series.

In the next theorem, the remainder term $\boldsymbol{R}_{n+1}(x,\theta(x))$ can be expressed in terms of θ, nth and $(n+1)$th derivatives of the function $f(x)$ at a point ξ lies between x_0 and x.

Theorem 2 [5]. *Assume that $f(x)$ and its derivatives up to order $n+1$ are continuous in a certain neighborhood of a point x_0. Suppose, furthermore, that x is any value of the argument value from the indicated neighborhood and θ is a restrictive Taylor's parameter. Then there is a point $\xi \in [x_0, x]$ such that the error of the approximation estimated by $\boldsymbol{R}_{n+1}(x,\theta(x))$ is given in the formula (5), for which*