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A class of semi-implicit predictor–corrector schemes for the time integration of atmospheric models

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ABSTRACT

In this paper a class of semi-implicit predictor–corrector time integration schemes is proposed. Linear stability analysis is used to identify promising methods and these are applied to the nonlinear system of the shallow water equations on an icosahedral grid. The model used is a testbed for the future development of a more complete atmospheric model. Experiments with standard test cases from the literature show that the investigated time integration schemes produce stable results with relatively long time-steps while maintaining a sufficient level of accuracy. These facts suggest that the analysed methods could be useful for the construction of a more complex model based on the Euler equations.

1. Introduction

The evolution of geophysical fluids is governed by time-dependent Partial Differential Equations (PDEs) which, for most practical cases, can be solved only with the use of approximate methods. The most general methodology employed to construct the appropriate approximations to these equations is based on semi-discretisation in space which leads to a large set of Ordinary Differential Equations (ODEs). There are a number of efficient and accurate methods used for spatial discretisation including spectral, finite difference, finite element and finite volume algorithms. The changing computer architectures and the increased role of massive parallelism has led in recent years to significant changes in the structure of the so-called dynamical cores of atmospheric models and consequently the entire models for the Numerical Weather Prediction (NWP). The increased role of finite element and finite volume schemes in accommodating the efficient partitioning of the computational domain, important for efficient calculations on massively parallel computers, is becoming particularly evident. The best examples are provided by several models constructed on the basis of Platonic polyhedra; see Staniforth and Thuburn [23] for a comprehensive review.

Formally we can write the system obtained after semi-discretisation of the PDEs as follows

$$\frac{d\psi}{dt} = F(\psi, t) + \mathcal{Q}(t)$$

where $\psi \in \mathbf{R}^d, F : \mathbf{R}^d \times [0, T] \to \mathbf{R}^d$, $\mathcal{Q}(t) \in \mathbf{R}^d$ represents sources and sinks, d is the number of degrees of freedom (depending on the spatial approximation technique, d could be interpreted as the number proportional to one of the following: finite elements, finite volumes, grid points or spectral modes) and $[0, T]$ is the time interval of the integration.

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The large set of ODEs obtained with one of the selected space discretisation techniques is characterised by extreme stiffness which reflects the existence of processes with different time-scales ranging from the very fast gravity and sound waves to the slow long waves governing large scale development of meteorological systems.

These wave motions are not the only reason for stiffness appearing in atmospheric models based on PDEs. Equally challenging are problems associated with nonlinear diffusion, particularly in the vertical direction, and chemical reactions (see Verwer et al. [24], for example). This fact shows that the main concerns of time integration raised in the context of design of dynamical cores are also important when coupling the model with parameterisation of physical processes such as the mixing in the planetary boundary layer.

The different time-scales represented in the semi-discrete system pose a significant challenge for prospective integration schemes. Implicit schemes which solve the problem of stiffness are often too expensive to implement because they require multiple evaluation and inversion of the Jacobian of the system. Explicit methods, on the other hand, require very short time-steps in order to maintain stability and so become prohibitively inefficient.

A closer analysis of the system obtained after semi-discretisation shows that, in many cases, we can write the general ODE system in the form

$$\frac{d\psi}{dt} = L\psi + N(\psi, t) + \mathcal{Q}(t)$$

where L is the linear operator represented by a sparse matrix and $N(\psi, t)$ is the nonlinear part of $F(\psi, t)$.

In the context of geophysical fluid dynamics, the linear operator represents the pressure gradient and divergence terms, which are crucial for the propagation of fast gravity waves, and the nonlinear operator represents the slower advective processes. The difference in the respective characteristic speeds suggests an optimum time integration strategy in the form of a semi-implicit method in which the linear part is treated implicitly and the nonlinear part explicity. In NWP, the pioneering semi-implicit method combined an implicit trapezoidal averaging with a centred, leapfrog discretisation for the explicit terms [16]. Since its introduction it has become the standard component of many atmospheric models.

Despite the enduring efficiency and simplicity of this approach, it has a number of known issues. In particular, the three time-level nature of the leapfrog scheme introduces a spurious computational mode. A time filter has been traditionally added to control this mode [2], although this is known to affect accuracy. Recently, a number of authors have proposed improved methods for the temporal discretisation in atmospheric models. Williams [25,26] has introduced a simple and effective modification to the Robert–Asselin filter for suppressing the computational mode; Amezcua et al. [1] found that this improved medium-range forecast skill in a primitive equation model. Moving beyond the leapfrog scheme, Giraldo [11] tested semi-implicit techniques based on backward differencing, and Durran and Blossey [7] have investigated a range of implicit–explicit linear multistep methods.

In this paper, we seek improved discretisation in time by considering multistage, predictor–corrector schemes with an implicit method applied at each stage. A number of explicit predictor–corrector schemes have been proposed for use in atmospheric models. For example, Kurihara [15] found that a leapfrog predictor step followed by a trapezoidal corrector successfully damped computational modes, while Kar [14] has suggested the combination of a second-order Adams–Bashforth predictor with a trapezoidal corrector. The semi-implicit multistage approach has also been tested: Kar [13], for instance, combined a third-order Runge–Kutta time differencing with a trapezoidal averaging of gravity wave terms at each iterative stage. Predictor–corrector methods have also been applied in semi-Lagrangian models [3,4].

The paper is organised as follows. In Sections 2 and 3 a class of schemes is introduced and a number of specific algorithms are evaluated with a simple linear system with two different frequencies. The most promising schemes are then selected to be used for the numerical integration of the shallow water equations in Section 4. The linear stability of these methods is first considered and the best are used for tests with a fully nonlinear icosahedral model. A number of standard test cases are used and the results are compared with benchmark runs using the explicit fourth-order Runge–Kutta algorithm. Characteristic of semi-implicit discretisations is the need to solve elliptic problems at each time-step. Due to the two stages in a predictor–corrector scheme, we need to solve two such problems per step. The issues of efficiency with the elliptic solver are explored in Section 5.

2. Semi-implicit predictor–corrector schemes

As discussed in the introduction, we consider a general differential equation in time where we have separated the fast, linear terms and the slower, residual nonlinears:

$$\frac{d\psi}{dt} = L\psi + N(\psi) \tag{1}$$

Note that in this work we ignore the source and sink terms, since the shallow water system to be solved is not forced.

The ‘traditional’ semi-implicit approach investigated by Kwizak and Robert [16] involves a leapfrog discretisation of the nonlinear terms with a trapezoidal treatment of the linear terms; we denote this scheme **SILF**:

$$\frac{\psi^{n+1} - \psi^{n-1}}{2\Delta t} = \frac{L\psi^{n+1} + L\psi^{n-1}}{2} + N(\psi^n) \tag{2}$$