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Pressure forcing and dispersion analysis for discontinuous Galerkin approximations to oceanic fluid flows

Robert L. Higdon *

Department of Mathematics, Oregon State University, Corvallis, OR 97331-4605, United States

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ABSTRACT

This paper is part of an effort to examine the application of discontinuous Galerkin (DG) methods to the numerical modeling of the general circulation of the ocean. One step performed here is to develop an integral weak formulation of the lateral pressure forcing that is suitable for usage with a DG method and with a generalized vertical coordinate that includes level, terrain-fitted, isopycnic, and hybrid coordinates as examples. This formulation is then tested, in special cases, with analyses of dispersion relations and numerical stability and with some computational experiments. These results suggest that the advantages of DG methods may significantly outweigh their disadvantages, in the settings tested here. This paper also outlines some other issues that need to be addressed in future work.

1. Introduction

The purpose of this paper is to derive and examine some properties of discontinuous Galerkin (DG) methods, as applied to the numerical modeling of ocean circulation.

Operational ocean models have traditionally used finite difference and finite volume methods on structured rectangular grids, although the idea of unstructured Voronoi grids is presently under development [26]. In practice, structured rectangular grids have usually been used with staggered arrangements of grid points known as the B-grid and the C-grid. Such grids have an advantage of simplicity, but they can allow troublesome grid noise and can give inaccurate propagation of inertia-gravity waves and/or Rossby waves, depending on the relation between the grid size and a length scale known the Rossby radius. More extensive discussions of these grids are included in Sections 6.2 and 6.3 and, for example, in [14,18].

For the class of DG methods, some often-quoted advantages include applicability to unstructured grids and the ability to attain high-order accuracy while maintaining high locality [7]. Some disadvantages, related to efficiency, include restrictive conditions on the maximum allowable time step for numerical stability [21] and the need to compute multiple degrees of freedom for each dependent variable.

The present work is the first part of an effort to examine the applicability of DG methods to ocean circulation modeling and to assess whether they have a net advantage over the methods that are used currently. Due to the overall complexity of this matter, the scope of the present paper is limited to the following goals:

(1) Formulate the pressure forcing in the partial differential equations that describe the conservation of momentum. Here, it is assumed that the vertical coordinate is a generalized coordinate that includes level, terrain-fitted, isopycnic, and hybrid coordinates as special cases. When a generalized vertical coordinate is used, the pressure forcing is the sum of

* Tel.: +1 541 737 5150; fax: +1 541 737 0517.
E-mail address: higdon@math.oregonstate.edu

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two terms, and in some circumstances the terms can have similar magnitude but opposite signs, so that their sum can be dominated by error. This representation can also be awkward for implementing in a weak form that is required for a Galerkin numerical method. Here, we go back to physical principles and derive, and then analyze, an integral weak form of the pressure forcing that avoids these difficulties.

(2) Analyze the accuracy of this approach by developing dispersion relations for the resulting DG spatial discretizations, and in particular compare the accuracy of such discretizations with finite difference approximations on the B-grid and C-grid. This methodology is then extended to give stability analyses of some time-stepping methods. In order to limit the complexity of the present paper, these analyses assume a reduced-dimension setting in which all flow variables are independent of one horizontal spatial variable. In this setting, the Coriolis parameter is nonzero and both components of velocity can be nonzero; in effect, we consider flow in an infinite straight channel in a rotating reference frame. The complexity is limited further by assuming linearized flow in a constant-density fluid. One conclusion is that the DG spatial discretizations can be much more accurate than the B- and C-grids. In particular, the DG formulation is not vulnerable to the problem of grid size versus Rossby radius for inertia-gravity waves that was mentioned above.

(3) Test the preceding ideas in some numerical computations. In one test problem described here, the higher spatial accuracy of the DG method can more than compensate for the restrictive bound on the time step and the need to compute multiple degrees of freedom. Another computation illustrates the well-balanced nature of the pressure forcing formulated here, in the constant-density case.

The DG method has been used extensively to solve the shallow water equations, which describe a single-layer (homogeneous) hydrostatic fluid (e.g., [12,22,25]). In addition, Kärnä et al. [20] have recently developed a DG model of three-dimensional coastal flows that uses a terrain-fitted vertical coordinate with a moving vertical mesh. Nair et al. [24] have developed a dynamical core for three-dimensional atmospheric circulation which uses a DG method on a cubed sphere for the horizontal discretization and a Lagrangian coordinate for the vertical discretization.

The purpose of the present paper is to develop a framework for a general vertical coordinate for ocean modeling and to perform the mathematical and computational analyses that are described above. An outline of this paper is the following.

In Section 2 we describe the equations for conservation of mass and momentum in terms of a generalized vertical coordinate, with two horizontal dimensions. In Section 3 we derive the weak forms of these equations for the reduced-dimension setting described above, with no assumption of linearity or constant density. This discussion emphasizes the formulation of the pressure term. The Appendix at the end of the paper gives the corresponding results for the general case of two horizontal orthogonal curvilinear coordinates on a rotating spheroid, with a generalized vertical coordinate.

Section 4 summarizes the remaining issues that will be discussed in the present paper, and it also gives an outline of other issues that are the subject of continuing work.

Section 5 describes the special case of a hydrostatic fluid of constant density, i.e., the shallow water equations. This discussion includes a detailed discussion of the pressure term, including Lax–Friedrichs interpolation to obtain pressure forcing at cell (element) edges, implementation with variable and discontinuous bottom topography, and a proof that the pressure forcing is well-balanced in this case.

Section 6 gives the analysis of numerical dispersion relations, and Section 7 gives analyses of some time-stepping methods. Some numerical computations are described in Section 8. Section 9 gives a summary.

2. Governing equations

The paper by Higdon [18] contains a derivation of the partial differential equations for conservation of mass, momentum, and tracers in a fluid that is in motion relative to a rotating spheroid. In that derivation the horizontal coordinates are arbitrary orthogonal curvilinear coordinates, and the vertical coordinate is a generalized coordinate, in a sense discussed in Section 2.1. Here we re-state the equations for conservation of mass and momentum. Curvilinear coordinates are used in the Appendix, but elsewhere in this paper the horizontal coordinates are taken to be rectangular Cartesian coordinates for the sake of notational simplicity.

In [18] it is assumed that the depth of the fluid is much smaller than the horizontal extent of the motions being studied. This shallow-water assumption implies that the fluid is very nearly in hydrostatic balance, i.e., vertical accelerations are small, and this condition will be assumed throughout the present paper.

2.1. Vertical coordinate

The partial differential equations that describe three-dimensional oceanic flows include a vertical coordinate, and in the numerical modeling of ocean circulation several such coordinates are in use. These include the following.

(i) The elevation z. This choice is the most traditional.

(ii) A terrain-fitted coordinate σ. This quantity has constant values at the top and bottom of the fluid, with a continuous transition between the top and bottom, and it is well-suited for representing bottom topography.