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A weak formulation for solving elliptic interface problems without body fitted grid

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ABSTRACT

A typical elliptic interface problem is casted as piecewise defined elliptic partial differential equations (PDE) in different regions which are coupled together with interface conditions, such as jumps in solution and flux across the interface. In many situations, such as the interface is moving, the challenge is how to solve such a problem accurately, robustly and efficiently without generating a body fitted mesh. The key issue is how to capture complex geometry of the interface and jump conditions across the interface effectively on a fixed mesh while the interface is not aligned with the mesh and the PDE is not valid across the interface. In this work we present a systematic formulation and further study of a second order accurate numerical method proposed in Hou and Liu (2005) [16] for elliptic interface problem. The key idea is to decompose the solution into two parts, a singular part and a regular part. The singular part captures the interface conditions while the regular part belongs to an appropriate space in the whole domain, which can be solved by a standard finite element formulation. In a general setup the two parts are coupled together. We give an explicit study of the construction of the singular part and the discretized system for the regular part. One key advantage of using weak formulation is that one can avoid assuming/using more regularity than necessary of the solution and the interface. We present the numerical algorithm and numerical tests in 3D to demonstrate the accuracy and other properties of our method.

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1. Introduction

Interface problems occur in many multi-physics and multi-phase applications in science and engineering. In this work we study a typical elliptic interface problem which is casted as piecewisely defined elliptic partial differential equations (PDE) in different regions which are coupled together with interface conditions, such as jumps in solution and flux across the interface. A natural numerical approach is to generate a mesh that fit the interface, i.e., a body fitted mesh that does not allow the interface to cut across a cell. Then each piecewise elliptic PDE and the jump conditions across the interface can be naturally put into a standard finite element formulation, e.g., [5]. However, in many situations, such as when the interface is moving, generating a body fitted mesh following a highly dynamically moving interface may be both computational expensive and challenging, especially in 3D. A more desirable approach is to solve the interface problem effectively on a fixed mesh which does not fit the interface in general. The key challenge is how to capture complex geometry of the interface and jump

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conditions across the interface to achieve both accuracy and robustness on a non body fitted mesh. Quite a few numerical methods in this category has been proposed and studied. For examples, immersed boundary method, immersed interface method, ghost fluid method, extended finite element method.

In [30,31], in order to simulate the flow patten of blood in the heart, Peskin proposed the "immersed boundary" method, which used numerical approximation of δ-function for singular sources on the interface. In [32], in order to compute two-phase flow, a level-set method was combined with the "immersed boundary" method. These methods are simple to use and but difficult to achieve high-order accuracy.

In [24,25], the solution is extended to a rectangular region by using Fredholm integral equations. The proposed method can deal with interface conditions $[u] \neq 0$ and $[u_n] = 0$ and when Greens function is available. The discrete Laplacian was evaluated using these jump conditions and a fast Poisson solver can be used to compute the extended solution. It can achieve second or higher-order accuracy.

A large class of finite difference methods have been proposed. The main idea is to use difference scheme and stencils carefully near the interface to incorporate jump conditions and achieve high order local truncation error using Taylor expansion. Using finite difference scheme typically requires taking high order derivatives of jump conditions and interface in Taylor expansion. Also property of the discretized linear system is hard to analyze for interface problem with general jump condition. The "immersed interface" method was proposed in [18]. This method incorporates the interface conditions into the finite difference scheme near the interface to achieve second-order accuracy based on a Taylor expansion in a local coordinate system. Second order differentiation of the interface is needed.

Various applications and extensions of the "immersed interface" method are provided in [20]. In [3], instead of using Taylor expansions, a variational method is used to define numerical stencils near the interface and a Lagrange multiplier approach is used to enforce jump conditions. A boundary condition capturing method based on dimension splitting, the ghost fluid method, was proposed in [21,9]. The method extends the solution from one side across the interface using the jump conditions. In [33], the boundary condition capturing method is improved with a multi-grid method. A weak formulation was used in [22] to prove the convergence. However, the method in [21] can only get first-order accuracy due to simple dimension by dimension extrapolation. It is in recent work [26] that for smooth interfaces the result was improved to second-order accuracy. Other dimension splitting type of methods based on finite difference include [40,6,34].

The existing finite element schemes on nonfitted meshes are usually designed by modifying the finite element basis near the interface. Examples are immersed finite element method [19,10], adaptive immersed interface method [4], extended finite element method [27,35,7]. The penalty finite element method [1,13] modifies the bilinear form near the interface by penalizing the jump of the solution value (with no general flux jump) across the interface. Recently a few unfitted mesh methods were developed based on the discontinuous Galerkin method using well known interior penalty technique to deal with jump and flux conditions for elliptic interface problem [23,2,12,36].

The Matched Interface and Boundary (MIB) method was developed in [40] and improved to handle sharp-edged interfaces in two dimension in [38] and in three dimension in [39]. Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [8,28] for two-dimensional problems and [29] for three-dimensional problems. Another recent work in this area is a class of kernel-free boundary integral (KFBI) methods for solving elliptic BVPs, presented in [37].

In this work we present a new formulation and further study of the second order accurate numerical method proposed in [16] for elliptic interface problem (1) with general matrix coefficient. The method can be viewed as a type of unfitted finite element method. The key idea in our formulation is to decompose the solution into two parts, a singular part and a regular part. The singular part is constructed explicitly in terms of the interface conditions and/or the regular part only locally near the interface. While the regular part lives in an appropriate space through the whole domain, which can be solved by a standard finite element formulation. In general, the coupling of singular and regular part leads to a non-symmetric discretized system. We give a explicit study of the construction of the singular part and the discretized system for the regular part in different setups. One key advantage of using weak formulation is that one can avoid assuming/using more regularity than necessary of the solution and the interface. Our method is also quite simple and easy to implement. The starting point is similar to that of the extended finite element method except that the interface conditions are enforced strongly for the local equations. Moreover, only standard finite element basis are introduced in our formulation. The method was also developed in [17] for sharp-edged interfaces, in [14] for elasticity interface problems and in [15] for multi-domain interface problems.

2. The equation and weak formulation

Consider an open bounded domain $\Omega \subset R^d$. Let Γ be an interface of co-dimension one, which divides Ω into disjoint open subdomains, Ω^- and Ω^+, hence $\Omega = \Omega^- \bigcup \Omega^+ \bigcup \Gamma$. Assume that the boundary $\partial\Omega$ and the boundary of each subdomain $\partial\Omega^\pm$ are Lipschitz continuous. Since $\partial\Omega^\pm$ are Lipschitz continuous, so is Γ. A unit normal vector of Γ can be defined a.e. on Γ, see Section 1.5 in [11].

We seek the solution to the following elliptic equation with piecewise smooth variable coefficient

$$-\bigtriangledown \cdot (\beta(x) \bigtriangledown u(x)) = f(x), \quad x \in \Omega \setminus \Gamma \tag{1}$$