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On the design of high order residual-based dissipation for unsteady compressible flows

A. Lerat *, K. Grimich, P. Cinnella

DynFluid Lab., Arts et Metiers ParisTech, 151 Boulevard de l'Hopital, 75013 Paris, France

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ABSTRACT

The numerical dissipation operator of residual-based compact (RBC) schemes of high accuracy is identified and analysed for hyperbolic systems of conservation laws. A necessary and sufficient condition (χ-criterion) is found that ensures dissipation in 2-D and 3-D for any order of the RBC scheme. Numerical applications of RBC schemes of order 3, 5 and 7 to a diagonal wave advection and to a converging cylindrical shock problem confirm the theoretical results.

1. Introduction

Classical methods for calculating compressible flows on a structured mesh rely on a directional approach in which space derivatives are approximated independently direction by direction. In the present paper, we study compact approximations that provide high accuracy not for each space derivative treated apart but for the complete residual r, i.e. the sum of all of the terms in the governing equations. For steady problems solved by time marching, r is the residual at steady state; it involves space derivatives only. For unsteady problems, r also includes the time derivative. Schemes of this type are referred-to as residual-based compact (RBC). They have been developed in the last ten years and applied to realistic flow configurations in aerodynamics and aeroacoustics (see [1–8]). Related schemes are the Residual Distribution Schemes (see [9–12]), developed on unstructured meshes, in which the residuals are distributed to the nodes of triangles or tetrahedrons following suitable design principles. A special feature of the RBC schemes is the use of a numerical dissipation term also constructed from the complete residual r. This unusual dissipation gives to the RBC schemes special properties that have not been fully analyzed so far. In practice, the RBC schemes are robust for steady flow computations, but some of them may have difficulties (described in Section 2.5) for unsteady problems, apparently due to a weak instability. Here, we present a comprehensive study of the residual-based dissipation term of high-order RBC schemes for the unsteady Euler equations. The study provides a deeper insight of the dissipation mechanism, provides a mathematical criterion (called χ-criterion) characterizing the dissipation for 2-D and 3-D problems and restores the stability of RBC schemes for unsteady problems. Given the importance of numerical dissipation in computational fluid dynamics, it is also hoped that the present work could help the development of other classes of high-order schemes.

The paper is organized as follows. Section 2 reminds the principles of high-order RBC schemes for solving a hyperbolic system of conservation laws on a 2-D Cartesian mesh. Such schemes are dissipative and compact (for instance, order 7 in

* Corresponding author.
E-mail address: alain.lerat@ensam.eu (A. Lerat).

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space is achieved using 5×5-points only). Compact schemes for compressible flows have been mainly developed as centered approximations in space (see [13–16] for instance) relying on the use of artificial viscosities, numerical filters or limiters for shock capturing. Upwind compact schemes have also been proposed in [17,18]. A peculiarity of the present RBC schemes is to use three independent compact approximations of the residual (four in 3-D). One applies to the usual residual at the current location j,k (main residual). The two others are involved in the dissipation and defined at location $j+\frac{1}{2},k$ or $j,k+\frac{1}{2}$ (mid-point residuals). A correct choice of the discretization of the mid-point residuals is essential to ensure dissipation for all flow conditions. This is why we try here to identify the effective dissipation operator, which is not obvious for a RBC scheme since the dissipation operator comes from high order expansions of the mid-point residuals. This work is done in two stages, in Section 3 and 4 respectively, for RBC schemes of order $2p-1=3,5$ and 7. In Section 5, the dissipation operator is cast into a general form (2-D partial differential operator of degree $2p$) and a necessary and sufficient condition, the χ-criterion, is found for this operator be always dissipative. Application of this criterion to the RBC schemes gives the correct coefficients to use in the high-order approximations of the mid-point residuals. A complete extension of the analysis to three space-dimension is presented in Section 6. Finally, numerical experiments are presented in Section 7 to confirm the relevance of the χ-criterion.

2. High-order RBC schemes

2.1. Concept of residual-based scheme

Let us consider an initial-value problem for the hyperbolic system of conservation laws:

$$w_t + f_x + g_y = 0 \tag{1}$$

where t is the time, x and y are Cartesian space coordinates, w is the state vector and $f=f(w), g=g(w)$ are flux components depending smoothly on w. The Jacobian matrices of the flux are denoted $A=\mathrm{d}f/\mathrm{d}w$ and $B=\mathrm{d}g/\mathrm{d}w$. System (1) is approximated in space on a uniform mesh $(x_j=j\delta x, y_k=k\delta y)$ with steps δx and δy of the same order of magnitude, say $\mathcal{O}(h)$, using a *residual-based* scheme. Such a scheme can be expressed only in terms of approximations of the exact residual, i.e. of the left-hand side of System (1). More precisely this scheme is a discrete form of

$$w_t + f_x + g_y = \frac{\delta x}{2}[\Phi_1(w_t + f_x + g_y)]_x + \frac{\delta y}{2}[\Phi_2(w_t + f_x + g_y)]_y \tag{2}$$

The coefficients Φ_1 and Φ_2 are numerical viscosity matrices that depend only on the eigensystems of the Jacobian matrices A and B and on the step ratio $\delta x/\delta y$. They use no tuning parameters or limiters. Their construction is presented in Section 2.4.

In System (2), the exact residual

$$r = w_t + f_x + g_y$$

is everywhere approximated in a space-centered way, but not at the same location or at the same order. The discrete form of (2) can be written as

$$\tilde{r}_{j,k} = \tilde{d}_{j,k} \tag{3}$$

where $\tilde{r}_{j,k}$ is a space-centered approximation of r, called the *main residual* and $\tilde{d}_{j,k}$ is the *residual-based dissipation* defined as:

$$\tilde{d}_{j,k} = \frac{1}{2}\left[(\Phi_1\tilde{r}_1)_{j+\frac{1}{2},k} - (\Phi_1\tilde{r}_1)_{j-\frac{1}{2},k}\right] + \frac{1}{2}\left[(\Phi_2\tilde{r}_2)_{j,k+\frac{1}{2}} - (\Phi_2\tilde{r}_2)_{j,k-\frac{1}{2}}\right] \tag{4}$$

where $(\tilde{r}_1)_{j+\frac{1}{2},k}$ and $(\tilde{r}_2)_{j,k+\frac{1}{2}}$ are space-centered approximations of r, called the *mid-point residuals* - see Fig. 1.

Despite appearance, the order of magnitude of the residual-based dissipation is not simply $\mathcal{O}(h)$ as it could seem from (2), but much smaller because the mid-point residuals approximate the exact residual which is everywhere null. Since centered differencing always leads to even order of accuracy, let the mid-point residuals be discretized so that

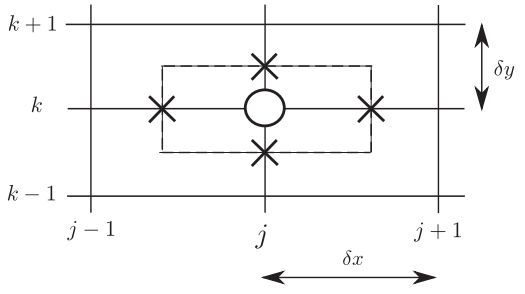

Fig. 1. Location of the discrete residuals, ∘: main residual $\tilde{r}$, ×: mid-point residual $\tilde{r}_1$ or $\tilde{r}_2$.