Journal of Computational Physics 235 (2013) 258–273

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Calculation of global optimal initial and boundary perturbations for the linearised incompressible Navier–Stokes equations

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ARTICLE INFO

Article history:
Received 8 February 2012
Received in revised form 19 October 2012
Accepted 28 October 2012
Available online 23 November 2012

Keywords:
Optimal perturbation
Adjoint equation
Optimisation

ABSTRACT

This study considers numerical methods for computation of optimal boundary and initial perturbations to incompressible flows. Similar to previous work, constrained Lagrangian functionals are built and gradient optimisation methods are applied to optimise perturbations that maximise the energy of perturbations in the computational domain at a given time horizon. Unlike most of the previous work in this field we consider both optimal initial and boundary condition problems and demonstrate how each can be transformed into an eigenvalue problem. It is demonstrated analytically and numerically that both optimisation and eigenvalue approaches converge to the same outcome, even though the optimisation approach may converge more slowly owing to the large number of inflection points. In a case study, these tools are used to calculate optimal initial and boundary perturbations to the Batchelor vortex. It is observed that when the flow is asymptotically stable, the optimal inflow perturbation is similar to the most unstable local eigenmode, while when the flow is stable/weakly unstable, the spatial distribution of the optimal inflow perturbation is similar to the local optimal initial perturbation.

1. Introduction

It is well-known that the evolution operators of perturbations in many open/closed flows are highly non-normal. The non-orthogonality of the eigenmodes of these operators can induce significant transient energy growth over short time intervals [1,2]. When the flow is asymptotically stable or weakly unstable, transient response to perturbations may induce significant change to the flow over finite time horizons.

Perturbations that maximise the energy in the response perturbation flow field are referred to as optimal. The optimal perturbation in the form of initial conditions has been extensively investigated in both local and global frameworks [2–4]. In steady or periodic base flows, the optimal initial perturbation is a linear combination of the eigenmodes of the evolution operator and over large time intervals, it approaches the leading adjoint eigenmode [5].

In local studies where the discretised matrix of the evolution operator is explicitly available, the optimal perturbation can be calculated through a singular value decomposition of the fully discretised operator, where the square of the largest singular value is the optimal energy growth and the corresponding right and left singular vectors are the optimal perturbation and its response, respectively [3].

Under more general conditions where the matrix form of the discretised operator is not available, commonly in global studies with large-dimensional base flows which are inhomogeneous in at least two directions, the optimal initial

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http://dx.doi.org/10.1016/j.jcp.2012.10.049

perturbation can be calculated through an iterative method, such as an eigenvalue method or an optimisation method. In the eigenvalue method, the optimal perturbation and energy growth are considered as the leading eigenvector/value of a joint operator, whose action is to integrate the perturbation forwards through the linearised evolution operator and then backwards through the adjoint operator [4,6–8]. Instead of discretizing this joint operator and building the matrix form of it directly, a Krylov sequence consisting of the iterative outcomes of this operator is constructed, and then an Arnoldi method or the Lanczos method is adopted to extract the leading eigenvector/eigenvalue of this joint operator. In this approach, the objective can be changed by merely changing the inner product weight in the definition of norms of the initial and final perturbations. [9] have used this eigenvalue approach to calculate the optimal upstream disturbance that experiences largest energy growth at a fixed downstream location in boundary layer flow. In the optimisation method, a constrained Lagrangian functional is built and the optimal perturbation is the velocity vector which maximises the Lagrangian functional [2,10,11]. This optimisation method has also been adopted to calculate the nonlinear optimal initial perturbations by [12,13]. In the latter work, the time-averaged dissipation, instead of the kinetic energy at a final time that is used commonly in the literature when defining an optimal perturbation, is chosen as the objective functional for better numerical convergence. These two approaches are discussed by [14] when solving optimal initial perturbations and optimal forcing localised in space in the Blasius boundary-layer flow and good agreement between the approaches is presented.

The complementary branch of the optimal perturbation problem —the optimal boundary perturbation problem — has received relatively limited attention. In local studies, some algorithms to calculate the boundary control velocity vectors have been proposed [10,11], but these algorithms are restrained to the context of parallel base flows and cannot be extended to global studies without significant modifications. Most recently [15] have proposed an optimisation algorithm to calculate the optimal spatial distribution of the inflow boundary perturbation to a stenotic flow, where the outflow velocity boundary condition is set to zero Dirichlet conditions, providing that the domain is adequately long.

In this work, we first outline the optimisation and eigenvalue methods for global initial perturbations, and then, develop an eigenvalue solver to compute the optimal boundary perturbations and demonstrate that this eigenvalue algorithm is equivalent to the reported optimisation approach [15]. A Robin outflow boundary condition is adopted to release the restriction on the size of the domain. Finally we implement these new numerical tools to calculate the optimal inflow boundary perturbation to a vortex flow as a case study.

2. Problem definition

Working from the incompressible Navier–Stokes equations

$$\partial_t \mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + Re^{-1}\nabla^2\mathbf{u}, \quad \text{with} \quad \nabla\cdot\mathbf{u} = 0,$$

where p is the modified or kinematic pressure, and $\mathbf{u}$ is the velocity vector, decomposing the flow field as the sum of a base flow and a perturbation i.e. $(\mathbf{u}, p) = (\mathbf{U}, P) + (\mathbf{u}', p')$ and omitting the interaction of perturbations, we obtain the linearised Navier–Stokes (LNS) equations, which govern the evolution of perturbations, as

$$\partial_t \mathbf{u}' = -\mathbf{U}\cdot\nabla\mathbf{u}' - (\nabla\mathbf{U})^T\cdot\mathbf{u}' - \nabla p' + Re^{-1}\nabla^2\mathbf{u}', \quad \text{with} \quad \nabla\cdot\mathbf{u}' = 0,$$

or more compactly, considering pressure is a dependent variable,

$$\partial_t \mathbf{u}' - L(\mathbf{u}') = 0. \tag{1}$$

If the base flow is homogeneous in the azimuthal direction in the cylindrical coordinates (z, r, θ), we can further decompose the perturbation field into azimuthal Fourier modes, each of which will evolve independently, such that:

$$(\mathbf{u}', p') = (u', v', w', p') = [u'_m(r,z), v'_m(r,z), w'_m(r,z), p'_m(r,z)]\exp(\mathrm{i}m\theta),$$

where m denotes an integer azimuthal wavenumber. If the base flow is homogeneous in the spanwise direction in a Cartesian coordinate, a similar spanwise Fourier mode can be defined at a real spanwise wavenumber. To keep notation reasonably compact in what follows we implicitly adopt Fourier decomposition for the perturbation field, only introduce its azimuthal/spanwise Fourier mode index m when required, and suppress representation of θ-dependence. In a similar vein, since we mainly consider perturbation fields in the following, the prime $(\prime)$ notation for perturbation variables is omitted hereafter.

We introduce scalar products defined on spatial domain Ω and its boundary $\partial\Omega$

$$(\mathbf{a},\mathbf{b}) = \int_\Omega \mathbf{a}\cdot\mathbf{b}\,\mathrm{d}V, \quad \langle\mathbf{a},\mathbf{b}\rangle = \int_0^\tau\int_\Omega \mathbf{a}\cdot\mathbf{b}\,\mathrm{d}V\mathrm{d}t,$$
$$[\mathbf{c},\mathbf{d}] = D\int_{\partial\Omega}\mathbf{c}\cdot\mathbf{d}\,\mathrm{d}S, \quad \{\mathbf{c},\mathbf{d}\} = D\int_0^\tau\int_{\partial\Omega}\mathbf{c}\cdot\mathbf{d}\,\mathrm{d}S\mathrm{d}t,$$

where $\mathbf{a},\mathbf{b}\in[0,\tau]\times\Omega$ and $\mathbf{c},\mathbf{d}\in[0,\tau]\times\partial\Omega$. D is a spatial length scale introduced into these definitions in order to maintain dimensional homogeneity as discussed below.

The optimal initial condition problem can be expressed as seeking an initial perturbation to maximise the energy growth over time interval τ, defined as the ratio of final energy at time τ and the initial energy at time 0 and denoted as G; the