Journal of Computational Physics 235 (2013) 497–514

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A minimum action method for small random perturbations of two-dimensional parallel shear flows

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ARTICLE INFO

Article history:
Received 30 March 2012
Received in revised form 5 October 2012
Accepted 8 October 2012
Available online 30 October 2012

Keywords:
Random perturbation
Dynamical system
White noise
Minimum action method
Rare events
Spectral elements
Finite elements

ABSTRACT

In this work, we develop a parallel minimum action method for small random perturbations of Navier–Stokes equations to solve the optimization problem given by the large deviation theory. The Freidlin–Wentzell action functional is discretized by *hp* finite elements in time direction and spectral methods in physical space. A simple diagonal preconditioner is constructed for the nonlinear conjugate gradient solver of the optimization problem. A hybrid parallel strategy based on MPI and OpenMP is developed to improve numerical efficiency. Both *h*- and *p*-convergence are obtained when the discretization error from physical space can be neglected. We also present preliminary results for the transition in two-dimensional Poiseuille flow from the base flow to a non-attenuated traveling wave.

1. Introduction

Dynamical systems are often subject to random perturbations since noise is ubiquitous in nature. Even when these random perturbations have a small amplitude, they can produce a profound effect on the long time dynamics by inducing rare but important events. A large number of interesting phenomena in physics, chemistry and biology such as phase transitions, biological switches and chemical reactions, etc., are examples of such noise-induced rare events [13].

When the random perturbations are small, the Freidlin–Wentzell theory of large deviations provides a rigorous mathematical framework for us to understand how the transitions occur and how frequent they are. The transition pathways between metastable sets in a dynamical system often have a rather deterministic nature. As the noise amplitude decreases to zero, the events for successful transitions between metastable sets have a sharply peaked probability around a certain deterministic path that is least unlikely. Special features of such a path tell us crucial information about the mechanism of the transition, which is closely related to the structure of the phase space. One class of examples that have been well studied for a long time are the gradient systems, for which the vector field is the gradient of a potential function. In gradient systems, the most probable transition path is the minimum energy path (MEP), which passes through the basin boundary between the stable states at some saddle points with one dimensional unstable manifold [16,21]. For non-gradient systems we need to consider the action functional instead of the energy, which is the central object to the Freidlin–Wentzell theory. The minimizer of the action functional provides the most probable transition path; the minimum of the action functional provides an estimate of the probability and the rate of occurrence of the transition. Thus an important practical task is to compute the minimum and minimizer of the action functional. A large number of algorithms have been designed for gradient systems.

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0021-9991/$ - see front matter
http://dx.doi.org/10.1016/j.jcp.2012.10.006

Some popular algorithms include the string method [3,5], nudged elastic band method [12], eigenvector- following-type method (e.g.[1]) as well as the dimer method [11], which usually take advantage of the fact that in gradient systems the transition paths are always parallel to the drift term of the stochastic differential equation. For general (non-gradient) systems, we need to minimize directly the Freidlin–Wentzell action functional and available algorithms include the minimum action method (MAM) [4], the adaptive MAM [19], the geometric MAM [10] and a high-order MAM [23].

In this work, we focus on the minimum action method for small random perturbations of Navier–Stokes equations. In particular, we are interested in whether the minimum action method can provide a new strategy to study the nonlinear instability of parallel shear flows. Stability of parallel shear flows, including plane Poiseuille flow, plane Couette flow, pipe flow, etc., is still a challenging dynamical problem, which people are not able to fully understand it through linear, (weakly) nonlinear and non-modal stability theories. In particular, the mechanism of the transition from the laminar flow to turbulence is still an open problem. Recently, people start to pay attention to study this problem by using an optimization strategy, where nonlinearity is included into the defined objective function. For example, in [15], the most dangerous initial perturbation leading to the turbulence state in plane Couette flow is examined by maximizing the time-averaged dissipation for a certain energy level of the initial disturbance. We look at this problem from a probabilistic point of view. Thinking of the Navier–Stokes equations perturbed by small noise, there exists a positive probability that a transition occurs from the laminar state to another state (including a turbulence state). By the Freidlin–Wentzell theory, an optimal path (or minimal action path) can be given by the minimizer of the Freidlin–Wentzell action functional, which describes the least unlikely route from the laminar state to the new state. We are interested in the intermediate states along this path. Since the minimal action path is closely related to the structure of the basin of attraction of the base flow [21,20,22], we expect to get useful information about the nonlinear instability along the minimal action path. As the first step of such a strategy, we develop the algorithm of the minimum action method for the stochastic Navier–Stokes equations.

This paper is organized as follows. We first describe the Freidlin–Wentzell theory for small random perturbations of dynamical systems in Section 2. A general methodology for the minimum action method of stochastic partial differential equations is given in 3. We develop the minimum action method for the stochastic Navier–Stokes equations in section 4. We include some numerical results for two-dimensional Poiseuille flows in Section 5, followed by a summary section.

2. Theoretical background

Although we are interested in the random perturbations of Navier–Stokes equations in this work, which are stochastic partial differential equations, we use stochastic differential equations to present the theoretical background of minimum action method for simplicity and without loss of generality. Let the random process $X_t = X(t) : \mathbb{R}_+ \to \mathbb{R}^n$ defined by the following stochastic ordinary differential equation (SODE):

$$dX_t = b(X_t)dt + \sqrt{\varepsilon}dW_t, \tag{1}$$

where W_t is a standard Wiener process in $\mathbb{R}^n$ and ε is a small positive parameter. Let $\phi(t) \in \mathbb{R}^n$ be an absolutely continuous function defined on $t \in [0,T]$. The Freidlin–Wentzell theory [7] tells us that the probability of $X(t)$ passing through the δ-tube about ϕ on $[0,T]$ is

$$\Pr(\rho(X,\phi) < \delta) \approx \exp\left(-\frac{1}{\varepsilon}S_T(\phi)\right), \tag{2}$$

where $\rho(\phi,\varphi) = \sup_{t\in[0,T]}|\phi(t) - \varphi(t)|$, and $S_T(\phi)$ is the action functional of ϕ on $[0,T]$, defined as

$$S_T(\phi) = \frac{1}{2}\int_0^T L(\dot{\phi},\phi)dt, \tag{3}$$

where $L(\dot{\phi},\phi) = |\dot{\phi} - b(\phi)|^2$. In general, we have the following large deviation principle

$$\lim_{\varepsilon\to 0}\varepsilon \log \Pr(X \in A) = -\min_{\phi\in A} S_T(\phi), \tag{4}$$

where A is a subset of the path space. Hence, in analogy with the Laplace's method, the basic contribution to $\Pr(X \in A)$ is given by the neighbourhood of the minimum of $S_T(\phi)$ when ε is small enough, in the sense that away from the minimizer of the action functional $S_T(\phi)$ the probability that the event A occurs through other possible choices will decay exponentially. The minimizer ϕ^*, which satisfies $S_T(\phi^*) = \min_{\phi\in A} S_T(\phi)$ is also called the "minimal action path" (MAP).

Different definitions of the set A in Eq. (4) correspond to many important phenomena that occur in dynamical systems. For example, if we are interested in the transition from one point a_1 to the other point a_2 in the phase space on the time interval $[0,T]$ due to small random perturbations, A can be defined as

$$A = \{\phi(t)|\phi(0) = a_1,\ \phi(T) = a_2\}.$$

The MAP will be the most probable path for the transition from a_1 to a_2 where the probability of the system taking all the other paths decays exponentially with respect to the noise amplitude ε according to the large deviation principle. Note that when a_1 and a_2 are attractors, it is more appropriate to define the set A as