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A space–time smooth artificial viscosity method for nonlinear conservation laws

J. Reisner [a], J. Serencsa [b,*], S. Shkoller [b]

[a] Los Alamos National Lab, XCP-4 MSF605, Los Alamos, NM 87544, USA
[b] Department of Mathematics, UC Davis, One Shields Ave., Davis, CA 95616, USA

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ABSTRACT

We introduce a new methodology for adding localized, space–time smooth, artificial viscosity to nonlinear systems of conservation laws which propagate shock waves, rarefactions, and contact discontinuities, which we call the *C*-method. We shall focus our attention on the compressible Euler equations in one space dimension. The novel feature of our approach involves the coupling of a linear scalar reaction–diffusion equation to our system of conservation laws, whose solution $C(x,t)$ is the coefficient to an additional (and artificial) term added to the flux, which determines the location, localization, and strength of the artificial viscosity. Near shock discontinuities, $C(x,t)$ is large and localized, and transitions smoothly in space–time to zero away from discontinuities. Our approach is a provably convergent, spacetime-regularized variant of the original idea of Richtmeyer and Von Neumann, and is provided at the level of the PDE, thus allowing a host of numerical discretization schemes to be employed.

We demonstrate the effectiveness of the *C*-method with three different numerical implementations and apply these to a collection of classical problems: the Sod shock-tube, the Osher–Shu shock-tube, the Woodward–Colella blast wave and the Leblanc shock-tube. First, we use a classical continuous finite-element implementation using second-order discretization in both space and time, FEM-C. Second, we use a simplified WENO scheme within our *C*-method framework, WENO-C. Third, we use WENO with the Lax–Friedrichs flux together with the *C*-equation, and call this WENO-LF-C. All three schemes yield higher-order discretization strategies, which provide sharp shock resolution with minimal overshoot and noise, and compare well with higher-order WENO schemes that employ approximate Riemann solvers, outperforming them for the difficult Leblanc shock tube experiment.

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1. Introduction

1.1. Smoothing conservation laws

The initial-value problem for a general nonlinear system of conservation laws can be written as an evolution equation,

$$\partial_t U(x,t) + \operatorname{div} F(U(x,t)) = 0 \text{ with } U|_{t=0} = U_0, \tag{1}$$

for an m-vector U defined on $(D+1)$-dimensional space–time. Such partial differential equations (PDE) are both ubiquitous and fundamental in science and engineering, and include the compressible Euler equations of gas dynamics, the

* Corresponding author. Tel.: +1 530 554 2632.
E-mail addresses: reisner@lanl.gov (J. Reisner), jserencs@math.ucsd.edu (J. Serencsa), shkoller@math.ucdavis.edu (S. Shkoller).

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magneto-hydrodynamic (MHD) equations modeling ionized plasma, the elasticity equations of solid mechanics, and numerous related physical systems which possess complicated nonlinear wave interactions.

It is well known that solutions of (1) can develop finite-time shocks, even when the initial data is smooth, in which case, discontinuities of U are propagated according to the so-called Rankine-Hugoniot conditions (see Section 2.1). It is important to develop stable and robust numerical algorithms which can approximate shock-wave solutions. Even in one-space dimension, nonlinear wave interaction such as two shock waves colliding, is a difficult problem when considering accuracy, stability and monotonicity. The challenge is maintaining higher-order accuracy away from the shock while approximating the discontinuity in an order-Δx smooth transition region where Δx denotes the spatial grid size.

As we describe below, a variety of clever discretization schemes have been developed and employed, particularly in one-space dimension, to approximate discontinuous solution profiles in an essentially non-oscillatory (ENO) fashion. These include, but are not limited to, total variation diminishing (TVD) schemes, flux-corrected transport (FCT) schemes, weighted essentially non-oscillatory (WENO) schemes, discontinuous Galerkin methods, artificial diffusion methods, exact and approximate Riemann solvers, and a host of variants and combinations of these techniques.

We develop a robust parabolic-type regularization of (1), which we refer to as the C-method, which couples a modified set of m equations for U with an additional linear scalar reaction–diffusion equation for a new scalar field $C(x,t)$. Thus, instead of (1) we consider a system of $m+1$ equations, which use the solution $C(x,t)$ as a coefficient in a carefully chosen modification of the flux. As we describe in detail below, the solution $C(x,t)$ is highly localized in regions of discontinuity, and transitions smoothly (in both x and t) to zero in regions wherein the solution is smooth. Further, as $\Delta x \to 0$, we recover the original hyperbolic nonlinear system of conservation laws (1).

1.2. Numerical discretization

In the case of 1-D gas dynamics, the construction of non-oscillatory, higher-order, numerical algorithms such as ENO by Harten et al. [1] and Shu and Osher [2,3]; WENO by Liu et al. [4] and Jiang and Shu [5]; MUSCL by Van Leer [6], Colella [7], and Huynh [8]; or PPM by Colella and Woodward [9] requires carefully chosen *reconstruction* and *numerical flux*.

Such numerical methods evolve cell-averaged quantities; to calculate an accurate approximation of the flux at cell-interfaces, these schemes reconstruct kth-order ($k \geqslant 2$) polynomial approximations of the solution (and hence the flux) from the computed cell-averages, and thus provide kth-order accuracy away from discontinuities. See, for example, the convergence plots of Greenough and Rider [10] and Liska and Wendroff [11]. Given a polynomial representation of the solution, a strategy is chosen to compute the most accurate cell-interface flux, and this is achieved by a variety of algorithms. Centered numerical fluxes, such as Lax–Friedrichs, add dissipation as a mechanism to preserve stability and monotonicity. On the other hand, *characteristic-type* upwinding based upon exact (Godunov) or approximate (Roe, Osher, HLL, HLLC) Riemann solvers, which preserve monotonicity without adding too much dissipation, tend to be rather complex and PDE-specific; moreover, for strong shocks, other techniques may be required to dampen post-shock oscillations or to yield entropy-satisfying approximations (see Quirk [12]). Again, we refer the reader to the papers [10,11] or Colella and Woodward [13] for a thorough overview, as well as a comparison of the effectiveness of a variety of competitive schemes.

Majda and Osher [14] have shown that *any* numerical scheme is *at best*, first-order accurate in the presence of shocks or discontinuities. The use of higher-order numerical schemes is, nevertheless, imperative for the elimination of error-terms in the Taylor expansion (in mesh-size) and the subsequent limiting of truncation error. Moreover, higher-order schemes tend to be less dissipative than there lower-order counterparts, as discussed by Greenough and Rider [10]; therein, a comparison between a 2nd-order PLMDE scheme and a 5th-order WENO scheme demonstrates the improved resolution of intricate fine structure afforded by 5th-order WENO, while simultaneously providing far less clipping of local extrema than PLMDE.

In multi-D, similar tools are required to obtain non-oscillatory numerical schemes, but the multi-dimensional analogues to those described above are generally limited by mesh considerations. For structured grids (such as products of uniform 1-D grids), dimensional splitting is commonly used, decomposing the problem into a sequence of 1-D problems. This technique is quite successful, but stringent mesh requirements prohibits its use on complex domains. Moreover, applications to PDE outside of variants of the Euler equations may be somewhat limited. For further discussion of the limitations of dimensional splitting, we refer the reader to Crandall and Majda [15], and Jiang and Tadmor [16]. For unstructured grids, dimensional splitting is not available and alternative approaches must be employed, necessitated by the lack of multi-D Riemann solvers. WENO schemes on unstructured triangular grids have been developed in Hu and Shu [17], but using simplified methods, which employ reduced characteristic decompositions, can lead to a loss of monotonicity and stability.

Algorithms that explicitly introduce diffusion provide a simple way to stabilize higher-order numerical schemes and subsequently remove non-physical oscillations near shocks. In the mathematical analysis of conservation laws (and in the truncation error of certain discretization schemes), the simplest parabolic-regularization is by the addition of a uniform linear viscosity. Choosing a constant $\beta > 0$, which depends upon mesh-size Δx and sometimes velocity or wave-speed, and adding

$$\beta(\Delta x)\partial_x^2 U(x,t) \tag{2}$$

to the right hand side of (1) provides a uniformly parabolic regularization of the hyperbolic conservation laws, and its discrete implementation smears sharp discontinuities across $O(\Delta x)$-regions and thus adds stabilization, but unfortunately, at the cost of accuracy. With the addition of uniform linear viscosity, shocks and discontinuities are captured in a non-oscillatory