Pattern Recognition Letters 111 (2018) 80–86

Contents lists available at ScienceDirect

Pattern Recognition Letters

journal homepage: www.elsevier.com/locate/patrec

Joint asymptotic normality of granulometric moments under multiple structuring elements

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ARTICLE INFO

Article history:
Received 8 August 2017
Available online 21 April 2018

Keywords:
Granulometry
Mathematical morphology
Random sets
Image processing

ABSTRACT

For a binary image composed of randomly sized disjoint grains, morphological granulometries produce a pattern spectrum containing textural information that can be used for image classification, image segmentation and model estimation. It has been shown that as the number of grains per image increases, any finite collection of granulometric moments based on the same structuring element are jointly asymptotically normal with a known analytic form for the moments. However, many applications require the use of multiple structuring elements, for instance horizontal and vertical linear generators, to draw out important textural characteristics. In this work, we prove the joint asymptotic normality of granulometric moments from multiple structuring elements. We also derive analytic expressions for the asymptotic mean vector and covariance matrix of the granulometric moments, including the previously unknown off-diagonal elements of the asymptotic covariance matrix corresponding to granulometric moments from distinct structuring elements.

1. Introduction

Mathematical morphology is a fundamental component of modern image processing, with a wide variety of real-world applications in numerous industries [1]. Though originally introduced in the1960s, there is still a thriving research community [2]. Here, we focus on binary morphological granulometries, which model sieving of random sets in granular or textural images [3]. In the simplest case, sieving is performed by a family of morphological openings by a compact, convex structuring element scaled by a continuous sizing parameter. The area of the image remaining after filtering decreases as the sizing parameter increases. This area can be used to construct a probability distribution function known as the *pattern spectrum* of the image relative to the structuring element. The pattern spectrum contains textural information about the image relative to the given structuring element. The moments of the pattern spectrum, or *granulometric moments*, are random variables employed as shape and texture signatures used in image classification, segmentation and model parameter estimation [4–11]. Consequently, the properties and statistics of granulometric moments are important.

Assuming disjoint grains, exact granulometric moments have been found in terms of model parameters [12] and are known to be marginally asymptotically normal relative to the number of grains in an image [10,13]. Later work extended the asymptotic theory to joint granulometric moments, showing that any finite length vector of granulometric moments from a single structuring element is asymptotically jointly normal [14]. Asymptotic normality justifies the use of classification methods for texture discrimination that either assume normality or are known to perform well under normal distributions, and analytic forms for the asymptotic mean and covariance can be used as prior knowledge to improve classification. However, many applications require granulometric moments from multiple generators, for example horizontal and vertical structuring elements.

In this work, we present a fundamental theorem extending the asymptotic theory to multiple generators. Our main contributions are twofold. First, we show that an arbitrary finite number of moments from an arbitrary finite number of structuring elements is asymptotically jointly normal, whereas prior work shows joint normality only under a single structuring element. Second, we provide, for the first time, a closed form equation for the asymptotic covariance between granulometric moments from distinct structuring elements. This equation also holds for the asymptotic covariance between granulometric moments from the same structuring element, and is consistent with equations derived in prior works for this special case. The asymptotic mean for any granulometric moment also falls out of the theory, and is consistent with derivations from prior works. The main theorem is then validated with a battery of analyses based on synthetically generated granulomet-

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https://doi.org/10.1016/j.patrec.2018.04.032
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ric images. We begin with an overview of classical granulometric mixing theory.

2. Granulometric mixing theory

Consider a set S representing grains or texture in an image. Let B be a compact convex set, and $t > 0$. The family of morphological openings of S by the homothetic structuring element tB is called a *granulometry*. The area of the opening $S \circ tB$ of S is known as the *size distribution* and denoted $\Omega(t)$. The area of S is $\Omega(0)$, and if S is a bounded set then $\Omega(t) = 0$ for sufficiently large t. The normalized size distribution $\Phi(t) = 1 - \Omega(t)/\Omega(0)$ is a valid cumulative distribution function called the *pattern spectrum* of S relative to B [3].

For a fixed image, S, and structuring element, B, the pattern spectrum $\Phi(t)$ has kth (non-central) moments $\mu^{(k)}(S, B)$, known as *granulometric moments*. Of particular importance are the *pattern spectrum mean (PSM)*, $\mu^{(1)}(S, B)$, and *pattern spectrum variance (PSV)*, $\mu^{(2)}(S, B) - (\mu^{(1)}(S, B))^2$. These granulometric moments can be extracted from granular or textural images as features for classification, clustering, or segmentation. If S is a random image, the granulometric moments PSM and PSV are themselves random variables, each having their own moments.

We assume the image is a mixture of different sized replicates of multiple primitives (grain types). In particular, let S be a random set composed of randomly sized, disjoint translates arising from d compact sets, $A_1, A_2, \ldots, A_d$:

$$S = \bigcup_{i=1}^{d} \bigcup_{j=1}^{n_i} (r_{ij} A_i + x_{ij}), \tag{1}$$

where i indexes the primitive, n_i is the number of grains of type i, and r_{ij} is the sizing variable for the jth grain of type i. The following theorem provides equations for the exact granulometric moments of this model assuming disjoint grains [12].

Theorem 1 (Granulometric mixing theorem)**.** *For the granulometry $\{S \circ tB\}$ generated by a convex, compact set (structuring element) B, the kth granulometric moment is given by*

$$\mu^{(k)}(S, B) = \frac{\sum_{i=1}^{d} \sum_{j=1}^{n_i} m[A_i] \mu^{(k)}(A_i, B) r_{ij}^{k+2}}{\sum_{i=1}^{d} \sum_{j=1}^{n_i} m[A_i] r_{ij}^{2}}, \tag{2}$$

where $m[A_i]$ is the area of A_i and $\mu^{(k)}(A_i, B)$ is the kth granulometric moment of A_i.

In practice, given S and B one computes $\mu^{(k)}(S, B)$ by actually performing the morphological opening to find $\Phi(t)$. The result would be theoretically equivalent to evaluating (2) were the radii for each grain, r_{ij}, known.

Since the grain radii are typically unknown and difficult to compute, in the rest of this paper we treat them as random variables. The following theorem, proved in [10] using a theorem by Cramér [15], shows that the granulometric moments are asymptotically normal as the number of grains increases under a certain probabilistic model for the grain radii, and provides the asymptotic moments. Define $\alpha_i = m[A_i] \mu^{(k)}(A_i, B)$, $\beta_i = m[A_i]$,

$$u_k = \frac{1}{n} \sum_{i=1}^{d} \sum_{j=1}^{n_i} \alpha_i r_{ij}^{k+2} \quad \text{and} \quad \nu = \frac{1}{n} \sum_{i=1}^{d} \sum_{j=1}^{n_i} \beta_i r_{ij}^{2}.$$

Also define $H(a, b) = a/b$. Then the kth granulometric moment of S based on structuring element B is

$$\mu^{(k)}(S, B) = H_k = H(u_k, \nu) = \frac{u_k}{\nu}.$$

Let EH_k, Eu_k and $E\nu$ denote the expected values of H_k, u_k and ν, respectively. For H at $(Eu_k, E\nu)$, define $H_{k,0} = H(Eu_k, E\nu)$, $H_{k,1} = \frac{\partial H}{\partial u_k}(Eu_k, E\nu)$ and $H_{k,2} = \frac{\partial H}{\partial \nu}(Eu_k, E\nu)$.

Theorem 2 (Asymptotic granulometric mixing theorem)**.** *Let the grain counts for each primitive, $n_1, n_2, \ldots, n_d$, occur in known fixed proportions $\theta_i = n_i/n$, $i = 1, 2, \ldots, d$, for total sample size $n = n_1 + n_2 + \cdots + n_d$. Suppose that*

1. *The random grain-sizing variables r_{ij} are independent, r_{ij} is selected from a sizing distribution, Π_i, and every r_{ij} has finite moments up to at least order $k + 2$.*
2. *There exists a constant, C, independent of n, and $q > 0$ such that $H \le Cn^q$ for $n > 1$.*
3. *H has first and second derivatives and its second derivatives are bounded by a constant D, independent of n, in a neighborhood of $(Eu, E\nu)$.*

Then, for the granulometry $\{S \circ tB\}$ generated by a convex, compact set B, the distribution of H_k is asymptotically normal with mean and variance given by

$$\begin{aligned} E[H_k] &= H_{k,0} + O(n^{-1}), \\ Var[H_k] &= H_{k,1}^2 Var[u_k] + 2 H_{k,1} H_{k,2} Cov[u_k, \nu] \\ &\quad + H_{k,2}^2 Var[\nu] + O(n^{-3/2}). \end{aligned}$$

Finally, the following theorem is proven in [14], which generalizes Theorem 2 to consider the joint distribution of multiple granulometric moments, but still only considers a single common structuring element, B. Equations for the asymptotic moments are available in [14].

Theorem 3. *Under the conditions of Theorem 2, any finite set of granulometric moments under a single structuring element is asymptotically jointly normal.*

3. Asymptotic joint normality under multiple structuring elements

Note that Theorem 2 from [10] and Theorem 3 from [14] only address granulometric moments under one structuring element. In particular, these works do not prove asymptotic joint normality under multiple structuring elements, and they do not provide equations for the asymptotic covariance between granulometric moments generated from distinct structuring elements. In this section, we present our main result, which extends Theorem 3 to include granulometric moments under multiple structuring elements. We begin by defining a vector $\mathbf{Z}$ that includes every combination of the first K order granulometric moments of image S under m structuring elements $(B_1, B_2, \ldots, B_m)$ as follows:

$$\begin{aligned} \mathbf{Z} = [&\mu^{(1)}(S, B_1), \cdots, \mu^{(1)}(S, B_m), \cdots, \\ &\mu^{(K)}(S, B_1), \cdots, \mu^{(K)}(S, B_m)]^T. \end{aligned} \tag{3}$$

Theorem 4. *Let image S be given by (1) with d primitives $(A_1, A_2, \ldots, A_d)$, and let the grain counts for each primitive, $n_1, n_2, \ldots, n_d$, occur in known fixed proportions, $\theta_i = n_i/n$ for $i = 1, 2, \ldots, d$, where $\theta_i \neq 0$ for each i. Also let the grain sizing variables, r_{ij}, be independent and drawn from a sizing distribution Π_i with finite moments up to at least order $2(K + 2)$, where Π_i is not a point mass for at least one i. Then $\mathbf{Z}$ is jointly asymptotically normal. The asymptotic mean of $\mu^{(k)}(S, B_l)$ is*

$$\bar{\mu}^{(k)}(S, B_l) = \frac{\sum_{i=1}^{d} \mu^{(k)}(A_i, B_l) \gamma_i \mu_{i,k+2}}{\sum_{i=1}^{d} \gamma_i \mu_{i,2}}, \tag{4}$$

where $\mu_{i,\,k}$ is the mean of r_{ij}^k and $\gamma_i = \theta_i m[A_i]$ for each i. The asymptotic covariance between $\mu^{(k)}(S, B_p)$ and $\mu^{(l)}(S, B_q)$ is

$$\begin{aligned} \frac{1}{n(\sum_{i=1}^{d} \gamma_i \mu_{i,2})^2} \sum_{i=1}^{d} \frac{\gamma_i^2}{\theta_i} \Big(&\bar{\mu}^{(k)}(S, B_p) \bar{\mu}^{(l)}(S, B_q) \sigma_{i,2,2}^2 \\ &- \bar{\mu}^{(k)}(S, B_p) \mu^{(l)}(A_i, B_q) \sigma_{i,2,l+2}^2 \end{aligned}$$