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Efficient multicut enumeration of k-out-of-n:F and consecutive k-out-of-n:F systems

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ABSTRACT

We study multiple simultaneous cut events for k-out-of-n:F and linear consecutive k-out-of-n:F systems in which each component has a constant failure probability. We list the multicuts of these systems and describe the structural differences between them. Our approach, based on combinatorial commutative algebra, allows complete enumeration of the sets of multicuts for both kinds of systems. We also analyze the computational issues of multicut enumeration and reliability computations.

A system S is formed by several components each of which can perform at different levels with different probabilities. The performance of the system is the result of the combination of the performance states of its components. A system is called *coherent* if the degradation of the performance of any of its components cannot lead to an improvement of the system's performance and the improvement of any component cannot lead to degradation of the system's performance. Coherent systems include most industrial and biological systems, networks or workflow diagrams. The reliability of a coherent system S is the probability that the system is performing at a certain working level. The unreliability of S is defined as the probability that it is performing below the required level. Here we focus on system's unreliability, but our methods can directly be applied to its reliability. The study of system (un)reliability is a central area in engineering. In the recent years new mathematical methods and approaches have been developed including combinatorial methods, the study of correlated probabilities or minimal path and cut enumeration, cf. the recent papers [4,7,8,14,22,24].

Let S be a coherent system with n components, each of which has a constant probability of failure p_i, $i \in \{1, \dots, n\}$. If all p_i's are equal and independent then the system is said to have i.i.d. (independent and identically distributed) components. We will consider that the failure probabilities of the components are mutually independent but do not need to be identical. The method does however apply to more general situations in which independency is not assumed. For the system S we define a set of *minimal* failure states, i.e. states under which the system is failed and such that the improvement of any of its components would lead the system to a working state. We call such minimal failure states *minimal cuts*, while any failure state is simply called a *cut*. The unreliability of S is described as the probability that at least one minimal failure event occurred.

We are interested in the behaviour of the system S under several simultaneous failure events. By studying the distribution of the variable $Y = \{\text{number of minimal failure events}\}$ we obtain further information about the structure and behaviour of S. This is relevant for the management of spare parts, supplies and maintenance of the system. The consideration of simultaneous failures is important for instance in high performance computational systems, see [23]. Analogous to cuts we define i-multicuts as those system states that

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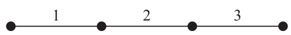
correspond to i simultaneous cuts. A minimal i-multicut is a state in which the system is suffering i simultaneous cuts but the improvement of any component makes the number of simultaneous cuts strictly smaller than i. Consider for instance the *series* system S with 3 components. A minimal cut of S is for instance {1} which means that the component 1 is failing. The cut given by {1, 2, 3} is a non-minimal cut, a non-minimal 2-multicut and a minimal 3-multicut for S.



The probability that a system S is suffering at least i simultaneous failures is the union event of all minimal i-multicuts of S. A first step to compute this probability is to enumerate the combinations of i simultaneous failures for each $i \geq 1$ and then compute the corresponding probabilities. This enumeration is hard in general for the number of minimal i-multicuts can grow exponentially.

k-out-of-n:F and consecutive k-out-of-n:F systems (Cons/k-out-of-n:F) are fundamental examples of coherent systems that generalize the basic series and parallel systems. These systems are ubiquitously studied and applied because of their redundancy, which is useful in a large variety of situations, see [6,9,11,25]. The tool we use to analyze multicuts and unreliability of those systems is the algebraic approach described by the authors in [18–21]. This approach consists in associating a monomial ideal I_S to the system S and analyze the features of S by studying the properties of I_S. The first section of the paper describes the algebraic approach to system reliability and its application to the problem of simultaneous failure events. In Sections 2 and 3 we focus on k-out-of-n:F systems and Cons/k-out-of-n:F systems, and give a description of their i-multicut structures, demonstrating their significant differences. Finally in Section 4 we consider some of the computational aspects of the problem and show the results of computer experiments based on the implementation of the tools previously described.

1. Algebraic reliability primer

Let S be a coherent system with n components. For ease of notation we assume that S is a binary system, i.e. the system and each of its components $c_1, \ldots, c_n$ can be either failed (1) or working (0). However, this method works also for multistate components. Let $\mathcal{F}_S$ be the set of failure states (cuts) of S and let $\overline{\mathcal{F}}_S$ be the set of minimal cuts. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbf{k}$. Each state of S corresponds to a monomial in R: the monomial $x_{i_1} \cdots x_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$ corresponds to the state in which components $c_{i_1}, \ldots, c_{i_r}$ are failed and the rest are working. The coherence property of S is equivalent to the fact that the elements of $\mathcal{F}_S$ correspond to the monomials in a monomial ideal, the *failure ideal* of S, which we denote by I_S. The unique minimal monomial generating set of I_S is formed by the monomials corresponding to the elements of $\overline{\mathcal{F}}_S$, see [18, Section 2]. Obtaining the set of minimal cuts of S amounts to compute the minimal generating set of I_S. In order to compute the unreliability of S we can use the numerator of the Hilbert series of I_S which gives a formula, in terms of $x_1, \ldots, x_n$ that enumerates all the monomials in I_S, i.e. the monomials corresponding to states in $\mathcal{F}_S$. Hence, computing the (numerator of the) Hilbert series of I_S provides a way to compute the unreliability of S by substituting x_i by p_i, the failure probability of the component i as explored in [18, Section 2]. Furthermore, in order to have a formula that can be truncated to obtain bounds for the reliability in the same way that the inclusion-exclusion formula is truncated to obtain the so-called Bonferroni bounds, we need a special way to write the numerator of the Hilbert series of I_S. This convenient form is given by the alternating sum of the ranks in any free resolution of I_S. Every monomial ideal has a *minimal* free resolution, which provides the tightest bounds among the aforementioned ones. In general, the closer the resolution is to the minimal one, the tighter the bounds obtained, see e.g. section 3 in [18].

In summary, the algebraic method for computing the unreliability of a coherent system S works as follows:

- Associate to the system S its failure ideal I_S and obtain the minimal generating set of I_S to get the set $\overline{F}_S$ of minimal cuts of S.
- Compute the Hilbert series of I_S to have the unreliability of S.
- Compute any free resolution of I_S. The alternating sum of the ranks of this resolution gives a formula for the Hilbert series of I_S, i.e. the unreliability of S which provides bounds by truncation at each summand. If the computed resolution is minimal, then these bounds are the tightest of this type.

Observe that this method works without alterations if the failure probabilities of the components of S are mutually different. For more details and the proofs of the results enunciated here, see [18]. For more applications of this method in reliability analysis, see [19–21].

When studying multiple simultaneous failures in a coherent system, the above methods can be extended to a filtration of ideals associated to the system under study as follows (cf.[16]): Let S be a coherent system in which several minimal failures can occur at the same time. Let Y be the number of such simultaneous failures. The event $\{Y \geq 1\}$ is the event that at least one elementary failure event occurs, which is the same as the event that the system fails. If x^α and x^β are the monomials corresponding to two elementary failure events, then $\mathrm{lcm}(x^\alpha, x^\beta) = x^{\alpha \vee \beta}$ corresponds to the intersection of the two events. The full event $\{Y \geq 2\}$ corresponds to the ideal generated by all such pairs. The argument extends to $\{Y \geq i\}$ and the study of the tail probabilities $\mathrm{prob}\{Y \geq i\}$. For each i, the set of i-multicuts generates an ideal, which altogether form a filtration. Let $I \subseteq \mathbf{k}[x_1, \ldots, x_n]$ be a monomial ideal and $\{m_1, \ldots, m_r\}$ its minimal monomial generating set. Let I_i be the ideal generated by the least common multiples of all sets of i distinct monomial generators of I, i.e. $I_i = \langle \mathrm{lcm}(\{m_j\}_{j\in\sigma}) : \sigma \subseteq \{1, \ldots, r\}, |\sigma| = i \rangle$. We call I_i the *i-fold* lcm-*ideal of* I. The ideals I_i form a descending filtration $I = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r$, which we call the lcm-*filtration* of I. If $I = I_S$ is the failure ideal of a system S then the minimal generating set of I_i is formed by the monomials corresponding to minimal i-multicuts of S. The survivor functions $F(i) = \mathrm{prob}\{Y \geq i\}$ for a coherent system, are obtained from the multigraded Hilbert function of the i-fold lcm-ideal I_i. The above considerations can be summarized in the following proposition.

Proposition 1.1. *Let Y be the number of failure events of a system S. If $\{m_1, \ldots, m_r\}$ is the set of monomial minimal generators of the failure ideal $I_S = I_1$ then $\mathrm{prob}\{Y \geq i\} = \mathrm{E}[\mathbf{1}_{M_i}(\alpha)] = \mathcal{H}_{I_i}(\mathbf{x})$, where $\mathbf{1}_{M_i}$ is the indicator function of the exponents of monomials in the i-fold* lcm-*ideal I_i and $\mathcal{H}_{I_i}(\mathbf{x})$ is the numerator of its multigraded Hilbert series.*

As a result, we also obtain identities and lower and upper bounds for multiple failure probabilities from free resolutions as in the single failure case.

2. Multiple simultaneous failures in k-out-of-n:F systems

A k-out-of-n:F system denoted by $S_{k,\,n}$ is a system with n components that fails whenever k of them fail. The failure ideal of $S_{k,\,n}$ is given by $I_{k,n} = \langle \prod_{i\in\sigma} x_i : \ \sigma \subseteq \{1, \ldots, n\}, |\sigma| = k \rangle$. Let $I_{k,\,n,\,i}$ be the i-fold lcm-ideal of $I_{k,\,n}$.

Theorem 2.1. *For all $k < j \leq n$ and $\binom{j-1}{k} < l \leq \binom{j}{k}$ we have that* $I_{k,n,l} = I_{j,n} = \langle \prod_{s\in\sigma} x_s : \ \sigma \subseteq \{1, \ldots, n\}, |\sigma| = j \rangle$