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Error analysis of octagonal distances defined by periodic neighborhood sequences for approximating Euclidean metrics in arbitrary dimension☆

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A B S T R A C T

In this paper, we consider approximation of Euclidean metrics by octagonal distances defined by periodic neighborhood sequences in arbitrary dimension. We derive an expression for maximum relative error (MRE) of an octagonal distance approximated by a weighted *t*-cost distance (WtD) function, with respect to the Euclidean metric in *n*-dimensional space. For this, we have used a general expression of MRE reported previously for a class of distances, in the form of a linear combination of weighted *t*-cost (WtD) and weighted (or chamfering) distances (CWD) and derived the expressions for specific cases of WtDs and CWDs. Further, this has also been applied to obtain theoretical expressions of MRE for *m*-neighbor distances (mND) in arbitrary dimension, and it improves the previously reported results regarding optimum value of *m* in an *n*-dimensional space. We also considered the adjustment of MRE values choosing an optimum scale factor. Computing theoretical values of scale adjusted MRE, we have reported good octagonal distances for approximating Euclidean metrics in different dimensional spaces. Previously, only a few such distances were reported for 2-D and 3-D spaces.

1. Introduction

Since the introduction of *octagonal distances* (OD) [21], they have drawn attention from several researchers [8,11,27,28] for various reasons. First and foremost, the shape of the disks in 2D is found to be in the form of an octagon, while disks of conventional cityblock and chessboard distances are diamonds and squares, respectively. The shape of an octagon is closer to the circular shape of a disk in the 2-D Euclidean space. Moreover, depending upon the underlying definition of a neighborhood sequence in constructing a path, features of octagons may vary. Exploiting their geometric properties, good octagonal distances were recommended previously [7,18] for approximating an Euclidean metric. The other interesting property of octagonal distances is that, they are truly digital distances assuming non-negative integral values only. Last but not the least, distance transforms of ODs can be computed using chamfering. This property makes them also alternative choices over some of the CWDs reported [2–5] as good candidates for approximating Euclidean distances.

From its early days, there have been continual efforts in finding good ODs for approximating Euclidean metrics [7,12,18]. However, the attempts were mostly restricted to 2D and 3D. In [21], it is shown that a generalized octagonal distance in 2D defined by a periodic neighborhood sequence of ω_1 cityblock motions followed by ω_2 chessboard motions, becomes a good candidate for approximating Euclidean metric, if $2\omega_2/\omega_1$ is as close as possible to $\sqrt{2}$. Later in an excellent treatise on the error analysis of simple octagonal distances in 2D, Das [12] provided theoretical expression of different error measures on approximating Euclidean metric, and suggested a list of good ODs in this regard. However, similar analysis in an arbitrary dimension (>2) is not yet reported, and remains an open problem (as commented in [12]) till date. This work attempts to fill that gap, though it is restricted to the analysis of maximum relative error (MRE)[1].

Following a different approach, called *geometric approach*, in [18] a list of 2D and 3D ODs is recommended considering the closeness of different attributes of digital circles and digital spheres of ODs to their Euclidean counter parts. Previous to this work Danielsson [7] also adopted a similar approach to report a few good ODs in 3D. However, till date no work on suggesting such distances in a dimension higher than three has been reported. In

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[1] Called *direct relative error* in [12].

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this work, we suggested a few such distances in higher dimensional spaces based on the error analysis. In our approach, we have exploited the geometry of hyperspheres in deriving the expression of MRE. The detailed analysis was already reported in [17], where an expression for MRE of a linear combination of WtD and CWD is derived. As ODs can very nearly be approximated by WtDs [14], the same analysis is also applicable in this case. In this work, specific simplified expressions of MREs for ODs defined by periodic neighborhood sequences, and CWDs are derived. They are used for obtaining good candidates of Euclidean approximation. We have also considered the fact that given a distance function, scaling may further improve the MRE value. For this we have theoretically derived an optimum scale factor for a distance function, and reported the scale adjusted MRE values.

Further, we have also applied this analysis for finding MRE values of m-neighbor distances (mND) [9], as these distances are special instances in the class of ODs. We obtained the theoretical expression for MRE for an mND in arbitrary dimension. This result is reported for the first time in this work. Earlier in [10], an error analysis on mNDs in arbitrary dimension was reported. In that work, though the authors did not provide any precise analytical expression of MRE, they reported its approximate bounds, and suggested a set of optimal mNDs in higher dimensional spaces ($\leq$40). Using the theoretical expression of MRE, we have obtained the same results in most cases reported by them [10]. Due to the preciseness of the derived expression, in some cases, we could better their estimates of the optimal m.

Though there is no prior work on error analysis of octagonal distances in dimensions higher than three, there are a few works reported for other distance families including their analysis in lower dimensional spaces such as in 2D, 3D and 4D. Borgefors carried out this analysis in 2D [2], 3D [3,4] and 4D [5] for CWDs, and suggested a few good distances in that family for approximating Euclidean metrics. In an interesting work [22], a distance function is proposed by constructing paths of varying neighborhoods with weighted distances among the neighbors. There are a few good distances in this class proposed for the purpose of approximation. However, this analysis was restricted to 2D. In a different algebraic form, [1] also studied a class of weighted distances, and provided an optimal distance for approximating Euclidean norms in arbitrary dimensions. In [16], following a geometric approach good CWDs are obtained in 2D, 3D and 4D. More recently linear combination of different digital distances [15] are explored for the same purpose in arbitrary dimension. In [15], we have shown how hyperspheres could be used for computing MREs and reported MREs of a few distances for approximating Euclidean metric in arbitrary dimensions ($\leq$10). The analysis is further extended for the linear combination of WtD and CWD (called WtCWD) in [17], and a very good approximation of Euclidean metric is reported in higher dimensional spaces ($\leq$100). In this work, we restrict ourselves to integral distance functions, involving ODs defined by periodic neighborhood sequences, and CWDs with integral weights. We may also note that for the purpose of Euclidean approximation, distances in a non-uniform grid in 2-D and 3-D [23–26] are also explored. In this study, we are excluding those classes of distances.

2. Octagonal distances (OD)

In this paper we consider only regular tessellations of n-D space. Let the space be represented by Z^n, where Z is the set of integers. A point $\underline{u} \in Z^n$ is also denoted as $\underline{u} = (u(1), u(2), \ldots, u(n))$. We also denote the zero vector $(0, 0, \ldots, 0)$ as $\underline{0}$. We should note here that for simplifying the notation, we consider only distances measured from the origin of the space, and denote a distance value at $\underline{u}$ as $d(\underline{u})$. As all the distance functions considered in this work are translation invariant, we can express the distance between two points $\underline{x}$ and $\underline{y}$ in this form (i.e. $d(\underline{y} - \underline{x})$) without loss of generality. We provide a brief review of the definitions and related properties of ODs below.

In Z^2, there are two types of neighborhood definitions, namely O(1) (or type-1) and O(2) (or type-2) neighbors. For type-1(2) neighbors, the coordinates can differ by unity at most in 1(2) position. The OD is defined by constructing paths between two points using an alternate sequence of type-1 and type-2 neighbors in 2D. It is the length of the shortest path between two points formed using such a predefined neighborhood sequence B. For a finite B, the sequence is extended periodically. For example, with $B = \{1, 1, 2\}$, the paths between any two points are formed using a sequence of neighborhood types of 1, 1, 2, 1, 1, ..., and so on. It can be shown that this distance is a metric, and its disks resemble the shape of an octagon. In [11], this concept is generalized to n-D by accommodating arbitrary sequence of m-neighborhoods ($1 \leq m \leq n$) in n-D.

As there are n neighborhood types in n-D, any arbitrary sequence of neighborhood types is denoted as $B = \{b(1), b(2) \ldots, b(m), \ldots\}$, where $\forall i, b(i) \in \{1, 2, \ldots, n\}$. As in 2D, in this case also, a finite B represents a cyclic sequence with a period $p = |B|$, and it is represented as $B = \{b(1), b(2), \ldots, b(p)\}$. From [8], we find that if B is a sorted sequence in nondecreasing order of the neighborhood types, the corresponding OD is a metric[2]. In this work, we consider only ODs defined by sorted neighborhood sequences[3]. For representing a sorted neighborhood sequence in n-D, we also use a fixed vector representation Ω of dimension n, such that the ith component ω_i denotes the number of times the type-i neighborhood occurs in the sequence B. For example in 3-D, $B = \{1, 1, 3, 3, 3\}$ can be equivalently represented by $\Omega = \{2, 0, 3\}$. With this representation, the length of the period of the sequence can be computed as $p = \sum_{i=0}^{n} \omega_i$. It may be noted that m-neighbor distances (mND) [9] are special cases of ODs, where the neighborhood sequence $B(=\{m\})$ consists of only the type-m neighborhood.

From [8], a closed form expression for an octagonal distance $OD^n(\underline{u}; B)$ is given below.

Lemma 1. *Given a neighborhood sequence $B = \{b(1), b(2), \ldots, b(p)\}$ in n-D, the generalized octagonal distance $OD^n(\underline{u}; B)$ is given by the following expression.*

$$OD^n(\underline{u}; B) = \max_{t=1}^{n} \left\{ p \left\lfloor \frac{D_t^n(\underline{u})}{f_t(p)} \right\rfloor + h(z_t; B_t) \right\} \tag{1}$$

where $B_t = \{b_t(1), b_t(2), \ldots, b_t(p)\}$ such that $b_t(i) = \min\{b(i), t\}$. Other terms are defined as follows:

$$f_t(j) = \begin{cases} 0 & j = 0, \\ \sum_{i=1}^{j} b_t(i), & \text{for } 1 \leq j \leq p. \end{cases} \tag{2}$$

$$z_t = D_t^n(\underline{u}) \bmod f_t(p), \tag{3}$$

and,

$$h(z_t; B_t) = \min\{k | f_t(k) \geq z_t\} \tag{4}$$

[2] There are other unsorted neighborhood sequences which may define a metric. For example, in [19], it is shown that in 2D, an OD defined by a neighborhood sequence becomes metric, if for any integer m, the sum of the first m values is not greater than the sum of m consecutive values anywhere in the sequence.

[3] Though we have reported our results with sorted sequences for enforcing the metricity properties of distance functions defined by them. Our analysis is also applicable if an OD defined by any arbitrary periodic neighborhood sequence is a metric.