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Average Consensus in Directed Networks of Multi-agents with Uncertain Time-varying Delays

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Abstract This paper investigates the average consensus problem in directed networks of multi-agent systems with uncertain time-varying delays. Fixed and switching topologies that are kept weakly connected and balanced are firstly analyzed. The original system is then transformed into a reduced dimension model. Based on Jensen′s inequality and reciprocally convex approach, sufficient conditions for average consensus are further presented. Specially, a less conservative upper bound of time-varying communication delays is derived in comparison with the existing results. Numerical examples confirm the effectiveness of the proposed method.

Key words Average consensus, multi-agent systems, uncertain time-varying, reciprocally convex

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The distributed cooperative control of multi-agent systems has attracted significant amount of interest from the academics and industry due to its various advantages such as lower operational costs, less system requirements, higher robustness, and stronger adaptivity. Some recent theoretical results and progresses are mainly reviewed[1], which are classified into four directions including consensus, formation control, optimization, and estimation.

We will focus here on the first direction (i.e., consensus). Consensus and the like (synchronization, rendezvous) refer to the group behavior that all of the agents asymptotically reach a certain common agreement through a distributed protocol, which is one of the most important issues of distributed multi-agent coordination. It is ubiquitous in nature such as the fish swimming, the birds migration, the bees and ants swarm collective behavior[2]. In control engineering fields such as distributed control of multiple vehicles, and cooperation of networked multi-robots, some consensus theories and algorithms have been studied[3−7].

The consensus of multi-agent systems crucially depends on network, topology and delay, where the network may be undirected or directed graph, the topology may be fixed or switching topology, and the delay may be constant or time varying. In fact, under the real network environment, due to the finiteness of signal transmission speed, communication time delay is inevitably introduced. It is well known that time delay may degrade the system performance or even cause the system instability. Therefore, some researches have reported on the consensus analysis of multi-agent systems with time delay to investigate its effect on system performance and stability. These results can be roughly classified into two categories according to whether the system is discrete-time or continuous-time.

For discrete-time systems, the works focus on the consensus problems for low-dimensional and high-dimensional multi-agent systems with time-varying delays. The state consensus problems for low-dimensional multi-agent systems with changing communication topologies and bounded time-varying communication delays are studied[8], where only instantaneous state information of every agent can be used and some sufficient conditions for state consensus of system are presented. However, for some agents, the state consensus cannot be guaranteed generally, if only delayed information of themselves can be provided. Using not only its own instantaneous state information of every agent but also its neighbors′ instantaneous state information[9], the consensus problems for discrete-time multi-agent systems with time-varying delays and switching interaction topologies are developed. For low-dimensional multi-agent systems, some researchers have investigated the consensus problems from other aspects in recent works[10−12]. For example, a new unified framework is established to deal with the consensus in directed network of discrete-time delayed multi-agent systems with fixed topology[11]. The consensus problem for discrete-time high-dimensional linear systems with or without delays is also researched[13].

For continuous-time systems, the consensus problems for second-order multi-agent systems have been researched in the recent years[14−20], where the quantized consensus, mean square average-consensus and impulsive consensus are investigated respectively[16−18]. For the first order multi-agent systems, some researchers consider the undirected graph with fixed topology or switching topology[21−24]. Theoretical framework for posing and solving consensus problem of undirected networks with fixed topology of strongly connected and balanced digraphs with communication time-delays is presented[21], and it is shown that the maximum time-delay is inversely proportional to the largest eigenvalue of the network topology or the maximum degree of the nodes of the network. In a directed graph with fixed topology or switching topology, the average consensus problem for system with constant and time-varying delays is studied in [25], and an upper bound of time-varying communication delay is obtained. Similar results can be found in [26−29], but H_∞ consensus problems in directed networks of agents with fixed and switching topologies are investigated[26] and the recent research of noisy links with times delay is presented[28]. Moreover, the average consensus problem is considered[29], where the system with switching topology and constant time delay is only considered. The consensus problem with dynamically changing topologies and nonuniform time-varying delays is researched, where one case of intermittent communication

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and data packet dropout is also considered[30]. These researchers focus on the directed graph with fixed topology or switching topology. In recent works, some researchers have investigated the consensus problems from various perspectives[31−34]. For example, the observer-based consensus of networked multi-agent systems with time-varying delays in a sampling setting is investigated[31]. System consensus problem in three cases is analyzed[33], where the system with fixed topology contains both communication and input time delays in each case. An observer-based control strategy for networked multi-agent systems with constant communication delay and stochastic switching topology is proposed[34]. However, how to get the formulation between maximum time-delay and the network topology for a directed graph with fixed topology or switching topology is still an open issue.

In this paper, the average consensus problem for continuous-time multi-agent systems in a directed network with uncertain time-varying delays is studied. We analyze fixed and switching topologies that are kept weakly connected and balanced. Firstly, based on a reduced dimension model, a Lyapunov-Krasovskii functional with uncertain time-varying delays is constructed. Then, using Jensen′s inequality and reciprocally convex approach, sufficient conditions for average consensus are obtained by a linear matrix inequality (LMI) set, and all the agents achieve the average consensus asymptotically. The main contribution of this paper is that the average consensus of multi-agent systems in the network is presented and a less conservative upper bound of time-varying communication delay is derived in comparison with the recent results.

The paper is organized as follows. Section 1 gives some preliminaries of graph theory. The average consensus problem is formulated in Section 2. Section 3 presents the main results of the paper. Simulation results are described in Section 4, followed by conclusions in Section 5.

1 Preliminaries of graph theory

Let $G(V,E,A)$ be a directed graph of order n, where the set of nodes $V=\{v_1,\cdots,v_n\}$, the set of edges $E\subseteq V\times V$. An edge of G is denoted by $e_{ij}=(v_i,v_j)$, where v_i is the tail of the edge and v_j is the head of the edge. The set of neighbors of node v_i is denoted by $N_i=\{v_j\in V|\,(v_j,v_i)\in E\}$. The node index of G belongs to a finite index set $I=\{1,2,\cdots,n\}$. $A=[a_{ij}]$ is a weighted adjacency matrix, where the adjacency elements are positive, i.e., $a_{ij}>0$, and $a_{ii}=0$, if and only of $v_j\in N_i$, $\forall i\in I$. The in-degree and out-degree of v_i are defined as

$$d_i(v_i)=\sum_{j=1}^{n}a_{ji}\ ,\quad d_o(v_i)=\sum_{j=1}^{n}a_{ij}$$

The degree matrix $D=[d_{ij}]_{n\times n}$ is a diagonal matrix with

$$D_{ij}=\begin{cases} d_o(v_i)=\sum\limits_{j=1}^{n}a_{ij}, & i=j\\ 0, & i\neq j\end{cases}$$

The Laplacian matrix of the graph G is defined as $L=D-A$. It is noted that every row sum of L is zero and $\mathbf{1}_n=[1,1,\cdots,1]^{\mathrm T}\in\mathbf{R}^n$ is an eigenvector of L associated with the eigenvalue $\lambda=0$. This therefore means that $\mathrm{rank}(L)\leq n-1$.

To derive the stability criteria, some lemmas and definitions are given firstly.

Lemma 1[21]**.** If the graph G is strongly connected, then its Laplacian L satisfies:

1) $\mathrm{rank}(L)=n-1$;

2) Zero is one eigenvalue of L, and $\mathbf{1}_n$ is the corresponding eigenvector, i.e., $L\mathbf{1}_n=0$;

3) The rest $n-1$ eigenvalues all have positive real parts. In particular, if the graph G is undirected, they are all positive and real.

Definition 1 (Balanced graph[21]). We say the node v_i of a graph $G(V,E,A)$ is balanced if and only if its in-degree and out-degree are equal, i.e., $d_o(v_i)=d_i(v_i)$. A graph $G(V,E,A)$ is called balanced if and only if all of its nodes are balanced. Obviously, any undirected graph is balanced.

Definition 2 (Balanced matrix[25]). A square matrix $M\in\mathbf{R}^{n\times n}$ is said to be a balanced matrix if and only if $\mathbf{1}_n^{\mathrm T}M=0$ and $M\mathbf{1}_n=0$.

Lemma 2[25]**.** Consider the matrix

$$A=\begin{bmatrix} n-1 & -1 & \cdots & -1\\ -1 & n-1 & \cdots & -1\\ \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & \cdots & n-1\end{bmatrix}$$

The following statements hold:

1) The eigenvalues of A are n with multiplicity $n-1$, and 0 with multiplicity 1. The vectors $\mathbf{1}_n^{\mathrm T}$ and $\mathbf{1}_n$ are left and the right eigenvectors of A associated with the zero eigenvalue, respectively.

2) There exists an orthogonal matrix O such that

$$O^{\mathrm T}AO=\begin{bmatrix} nI_{n-1} & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}$$

and O is the matrix of eigenvectors of A. For any balanced matrix $B\in\mathbf{R}^{n\times n}$,

$$O^{\mathrm T}BO=\begin{bmatrix} * & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}$$

Remark 1. It is obvious that when the graph G is strongly connected, its Laplacian L is a balanced matrix. According to Lemmas 1 and 2, the following equation holds:

$$U^{\mathrm T}LU=\begin{bmatrix} {U_1}^{\mathrm T}LU_1 & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}=\begin{bmatrix} \tilde{L} & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}$$

where $U=[U_1,\mathbf{1}_n/\sqrt{n}]$ is an orthogonal matrix of eigenvectors of L, and $U_1\in\mathbf{R}^{n\times(n-1)}$ is the first $n-1$ columns of U.

Lemma 3[35]**.** Let G be a balanced digraph, then G is strongly connected if and only if G is weakly connected.

Remark 2. The requirement of graph G that we discuss is strongly connected in the above, but we can obtain that it should be weakly connected by Lemma 3.

Definition 3[36]**.** Let $\Phi_1,\Phi_2,\cdots,\Phi_N:\mathbf{R}^m\mapsto\mathbf{R}^n$ be a given finite number of functions such that they have positive values in an open subset D of $\mathbf{R}^m$. Then, a reciprocally convex combination of these functions over D is a function of the form

$$\frac{1}{\alpha_1}\Phi_1+\frac{1}{\alpha_2}\Phi_2+\cdots+\frac{1}{\alpha_N}\Phi_N:D\mapsto\mathbf{R}^n$$

where the real numbers α_i satisfy $\alpha_i>0$ and $\sum_i\alpha_i=1$.